On homogeneous controllability functions

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The controllability function method, introduced by V. I. Korobov in late 1970s, is known to be an efficient tool for control systems design. This paper bridges the method with the homogeneity theory popular today. In particular, it is shown that the so-called homogeneous norm is a controllability function of the system in some cases. Moreover, the closed-loop control system is homogeneous in a generalized sense. This immediately yields many useful properties of the system such as robustness (Input-to-State Stability) with respect to a rather large class of perturbations.

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1. Introduction

Most of physical systems are symmetric in the view of Noether’s Theorem [23]. Differential symmetries can discovered in both ODE (Ordinary Differential Equation) and PDE (Partial Differential Equation) models of dynamical systems [9], [8], [27]. One of Lie symmetries, that is rather popular in control theory, is homogeneity. The homogeneity is a dilation symmetry known since 18th century, when Leonhard Euler studied functions $x \to f(x)$ which are symmetric with respect to uniform dilation $x \to \lambda x$ of its argument, namely, $f(\lambda x) = \lambda^\nu f(x)$, $\forall \lambda > 0, \forall x$, where $\nu$ is a real number. Such functions were called homogeneous and the number $\nu$ was referred as the homogeneity degree. It seems that a generalized homogeneity (the symmetry with respect to a non-uniform dilation) was first studied by Vladimir Zubov in [36]. The homogeneity is useful for analysis of nonlinear finite-dimensional dynamical systems (see also [36], [11], [13] [14], [33], [4], [2]) as well as non-linear controllers/observers design (see [13], [7], [10], [1], [19]). Homogeneity degree specifies a convergence rate of any asymptotically stable homogeneous system (see e.g. [22]). Homogeneous approximations of nonlinear models are useful in the case when a linear approximation is not informative or simply impossible [1]. Homogeneous control systems have similar properties to linear ones, e.g., local stability yields global stability, invariance of the compact set is equivalent to stability, etc. However, they may have better control quality, e.g., faster convergence, better robustness and less overshoot. For more details we refer the reader to [27, Chapter 1].

The controllability function method [16] is an efficient tool for control systems design. The monograph [17] presents most detailed study of the method and its applications to different control problems. Formally, a controllability function has no relation with the homogeneity theory. However, in some cases, its design is implicitly inspired by the homogeneity and uses the dilation symmetry of the system. This paper bridges controllability function method with the homogeneity theory showing that the control/controllability function design for linear plants results in a generalized homogeneous control system, that inherits all good properties of homogeneity, e.g., robustness (Input-to-State Stability) with respect to a sufficiently large class of perturbations.

The paper is organized as follows. First, the controllability function method and elements of the homogeneity theory are remained. Next, the main theorem about homogeneous controllability functions for linear plants is proven. Finally, some remarks and conclusions are given.

2. On controllability function method

Following [17] the controllability function $\Theta : \mathbb{R}^n \rightarrow [0, +\infty)$ is a Lyapunov-like function of a closed-loop system $u = u(x)$ realizing a position control in a finite time. More precisely, the properties of $\Theta$ are characterized by the following theorem.
Theorem 1. [17, Theorem 1.2, page 19] Let us consider the controlled process
\[ \dot{x} = f(x,u), \quad t > 0, \quad f : \mathbb{R}^{n+m} \to \mathbb{R}^m \] (1)
where \( x(t) \in \mathbb{R}^n, u \in \Omega \subset \mathbb{R}^m \) and the function \( f \) is locally Lipschitz continuous on \((\mathbb{R}^n \setminus \{0\}) \times \Omega\), i.e.,
\[ \| f(x_1, u_1) - f(x_2, u_2) \| \leq L(\rho_1, \rho_2)(\|x_1 - x_2\| + \|u_1 - u_2\|), \quad x_i \in K(\rho_1, \rho_2), \|u_i\| \in \Omega, i = 1, 2, \]
where \( K(\rho_1, \rho_2) := \{ x : 0 < \rho_1 \leq \|x\| \leq \rho_2 \} \).

Let there exist a function \( \Theta : \mathbb{R}^n \to [0, +\infty) \) such that
1) \( \Theta > 0 \) for \( x \neq 0 \) and \( \Theta(0) = 0 \);
2) there exists \( c > 0 \) such that the set is \( Q := \{ x : \Theta(x) \leq c \} \) is bounded;
3) there exists a function \( u : Q \to \Omega \) satisfying the inequality
\[ \frac{\partial \Theta(x)}{\partial x} f(x,u(x)) \leq -q \Theta^{1-\frac{1}{\alpha}}(x), \quad \forall x \in Q \setminus \{0\} \] (2)
for some \( q > 0 \) and \( \alpha > 0 \), such that \( u \) satisfies the Lipschitz condition on
on any set \( K(\rho_1, \rho_2) \cap Q \).

Then the trajectory \( x(t) \) of the closed-loop system (1) with \( x(0) = x_0 \in Q \) reaches the state \( x = 0 \) at an instant of time \( T(x_0) \leq \frac{\alpha}{q} \Theta^\frac{1}{\alpha}(x_0) \). Moreover, if \( \alpha = \infty \) then \( x(t) \to 0 \) as \( t \to \infty \).

If \( \alpha = 1 \) and the symbol \( \leq \) in (2) is replaced with \( = \), then the inequality becomes the Bellman (dynamic programming) equation and the function \( \Theta \) is the settling time function of the closed-loop system. On the other hand, the controllability function \( \Theta \) satisfies the classical [20] (resp., a generalized [32], [5]) Lyapunov theorem for \( \alpha = +\infty \) (resp. \( 0 < \alpha < +\infty \)).

Example 1. [17, page 21] A controllability function \( \Theta \) for the chain of integrators
\[ \dot{x} = Ax + Bu, \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \] (3)
can be designed implicitly as a solution of the following algebraic equation
\[ a_0 \Theta = x^\top D(\Theta)FD(\Theta)x, \quad x \neq 0 \] (4)
where \( a_0 > 0 \), a symmetric matrix \( F = F^\top \in \mathbb{R}^{n \times n} \) satisfies\(^1\) the linear matrix inequalities (LMIs)
\[ F(A + BK) + (A + BK)^\top F \preceq 0, \quad F > 0 \] (5)
\(^1\)The sign \( \prec \) (resp. \( \succ \)) denotes the negative (resp. positive) definiteness of a symmetric matrix.
with $K \in \mathbb{R}^{1 \times n}: A + BK$ - Hurwitz, and the matrix-valued function $D$ is defined as follows

$$D(\Theta) = \text{diag}\left(\Theta - \frac{m+n-2+i+1}{2\alpha}\right)$$

(6)

with $\alpha$ and $m$ are natural numbers being selected such that the linear matrix inequality (LMI)

$$F - H_\alpha F - H_\alpha F \succ 0, \quad H_\alpha = \text{diag}\left(-\frac{m+n-2+i+1}{2\alpha}\right),$$

(7)

is fulfilled. The corresponding controller is given by

$$u(x) = \Theta^{m-n-1}K D(\Theta)x.$$  

(8)

3. Homogeneity

This section presents a brief summary of the homogeneity theory of dynamical system in $\mathbb{R}^n$.

3.1. Dilations in $\mathbb{R}^n$

The standard (Euler’s) dilation in a vector space is the operator $x \rightarrow \lambda x$, $\lambda > 0$. Generalized dilations and the generalized homogeneity in $\mathbb{R}^n$ are introduced in [36], [15], [14], [26].

**Definition 1.** [14] A family of mappings $d(s) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $s \in \mathbb{R}$ is a said to be a dilation group (or, simply, a dilation) in $\mathbb{R}^n$ if

- (Group property) $d(0)x = x$, $d(t) \circ d(s)x = d(t+s)x$, $\forall t, s \in \mathbb{R}, \forall x \in \mathbb{R}^n$;

- (Limit property) $\lim_{s \rightarrow -\infty} \|d(s)x\| = 0$ and $\lim_{s \rightarrow +\infty} \|d(s)x\| = \infty$ for all $x \neq 0$.

Below we deal only with the so-called linear dilation [26] defined as follows:

$$d(s) := e^{sG_d} = \sum_{i=0}^{\infty} \frac{s^i G_d^i}{i!},$$

(9)

where $G_d \in \mathbb{R}^{n \times n}$ is an anti-Hurwitz matrix being a generator of the dilation. For $G_d = I_n$ (the identity matrix), we have the standard (Euler’s) dilation $d(s) = e^{sI_n}$ but the case of diagonal matrix $G_d$ corresponds to the so-called weighted dilation [36], [31], [12], [18], [1], [28] popular today.

**Definition 2.** [26] A linear dilation $d$ is said to be monotone with respect to a norm $\| \cdot \|$ if there exists $\beta > 0$ such that

$$\|d(s)\| \leq e^{\beta s}, \quad \forall s \leq 0,$$

(10)

where $\|d(s)\| = \max_{x \neq 0} \frac{\|d(s)x\|}{\|x\|}$ is the matrix norm of $d(s)$.
A criterion of the monotonicity of the linear dilation in $\mathbb{R}^n$ is given by the following lemma.

**Lemma 1.** [26] Any linear dilation $d$ in the finite dimensional space $\mathbb{R}^n$ is monotone with respect to the weighted Euclidean norm $\|x\|_P = \sqrt{x^\top P x}$, $x \in \mathbb{R}^n$ provided that the symmetric matrix $P = P^\top \in \mathbb{R}^{n \times n}$ satisfies the linear matrix inequality

$$G_d^\top P + PG_d \succ 0, \quad P \succ 0,$$

(11)

where the symbol $\succ 0$ means that the matrix is positive definite.

It is known [26] that any vector from $\mathbb{R}^n \setminus \{0\}$ can be uniquely projected to a unit sphere using monotone dilation. More precisely, for any $x \in \mathbb{R}^n \setminus \{0\}$ there exists a unique pair $(z, s) : \|z\| = 1, s \in \mathbb{R}$ such that $d(s)x = z$. We use this feature of monotone dilation in our further constructions.

### 3.2. Canonical homogeneous norm

Any compact and convex neighborhood of the origin induces a norm in $\mathbb{R}^n$ by means of the standard dilation $x \rightarrow \lambda x, \lambda > 0$ (see, e.g., [30]). In the general case, the similar construction for monotone dilation leads to the so-called homogeneous norm [10].

**Definition 3.** The function $\|\cdot\|_d : \mathbb{R}^n \rightarrow [0, +\infty)$ defined as follows $\|0\|_d = 0$ and

$$\|x\|_d = e^{sx} : \|d(-sx)x\| = 1, \quad x \neq 0,$$

is called the canonical homogeneous norm, where $d$ is a monotone dilation with respect to the norm $\|\cdot\|$ in $\mathbb{R}^n$.

The canonical homogeneous norm is continuous on $\mathbb{R}^n$ and locally Lipschitz continuous on $\mathbb{R}^n \setminus \{0\}$ (see, [26]). Moreover, in [27, Theorem 7.1., page 188] it is proven that $\|\cdot\|_d$ is, in fact, a norm (in the classical sense) in a finite-dimensional space $\tilde{\mathbb{R}}^n$ homeomorphic\(^2\) to $\mathbb{R}^n$.

Below we show that the canonical homogeneous norm is the controllability function for linear control systems considered in [17, Chapter 1]. The related analysis would require differentiability of $\|\cdot\|_d$ on $\mathbb{R}^n \setminus \{0\}$. The latter can always be guaranteed for a canonical homogeneous norm induced by a weighted Euclidean norm.

**Lemma 2.** [26] If the canonical homogeneous norm $\|\cdot\|_d$ is induced by the norm $\|x\| = \sqrt{x^\top P x}$ with a symmetric matrix $P = P^\top \in \mathbb{R}^{n \times n}$ satisfying (11) then $\|\cdot\|_d \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ and

$$\frac{\partial}{\partial x} \|x\|_d = \|x\|_d x^\top d^\top (\ln \|x\|_d) P d (\ln \|x\|_d) x, \quad x \neq 0.$$  

(12)

\(^2\)The space $\tilde{\mathbb{R}}^n$ consists of vectors from $\mathbb{R}^n$, but a sum of vectors and a multiplication of a vector by a scalar are defined differently.
Below we show that a properly designed canonical homogeneous norm is a controllability function for linear control plant.

3.3. Homogeneous systems. By definition, we say that an ODE is homogeneous if its right-hand side is a homogeneous vector field.

**Definition 4.** [14] A vector field (resp. a function) $f : \mathbb{R}^n \to \mathbb{R}^n$ (resp. $h : \mathbb{R}^n \to \mathbb{R}$) is said to be $d$-homogeneous of the degree $\nu \in \mathbb{R}$ is

$$f(d(s)x) = e^{\nu s} d(s) f(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n,$$

(resp. $h(d(s)x) = e^{\nu s} h(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n$),

where $d$ is a dilation in $\mathbb{R}^n$.

The canonical homogeneous norm is the simplest example of the homogeneous function of the degree 1.

**Example 2:** The system

$$\dot{x} = f(x) := \begin{pmatrix} x_2 \\ -k_1 x_1^{\frac{5}{3}} - k_2 x_2^{\frac{1}{2}} \end{pmatrix}$$

is $d$-homogeneous of the degree $\nu = -2$ with respect to the dilation

$$d(s) = \begin{pmatrix} e^{5s} & 0 \\ 0 & e^{3s} \end{pmatrix}.$$

Indeed, simple computations show

$$f(d(s)x) = e^{-2s} d(s) f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}.$$

Moreover, one can be shown [29] that

$$V(x) = c \left( \frac{5k_1 |x_1|^{\frac{2}{3}}}{6} + \frac{x_2^2}{2} \right)^{\frac{4}{3}} + k_1 x_1 x_2, \quad c > \max \left\{ \left( \frac{3}{4} \right)^{\frac{4}{3}}, k_1^{\frac{5}{2}} 2^{\frac{4}{3}} \right\}$$

is the controllability function (in the sense of Theorem 1)

$$\exists q > 0 : \dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -q V^{\frac{4}{3}}(x), \quad \forall x \neq 0,$$

of the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

with the feedback control

$$u(x) = -k_1 x_1^{\frac{1}{3}} - k_2 x_2^{\frac{1}{2}}, \quad k_1 > \frac{5}{6} k_2.$$

To highlight some features the homogeneous systems we present the series of known results.
Theorem 2. [36] If \( x(t, x_0) \) is a solution of a \( d \)-homogeneous system
\[
\dot{x} = f(x), \quad t > 0, \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad f \in C(\mathbb{R}^n)
\] with the initial condition \( x(0) = x_0 \) then
\[
x(t, d(s)) = d(x(e^{\mu s}t, x_0))
\]
is a solution of the same ODE with the scaled initial condition \( x(0) = d(s)x_0 \).

The latter theorem implies that any local property of the system (14) can be extended globally. For instance, local stability implies global stability, global-in-time existence of solutions for small initial data yields global-in-time existence of solutions for large initial data, etc.

The Zubov-Rosier Theorem [36], [31] given below is the converse Lyapunov theorem for homogeneous systems. For shortness, we say that a "system is stable" if its zero solution is stable.

Theorem 3. [31] A \( d \)-homogeneous system (14) is asymptotically stable if and only if there exists a positive definite \( d \)-homogeneous function \( V : \mathbb{R}^n \rightarrow [0, +\infty) \) of degree 1 such that \( V \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}) \) and
\[
\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -qV^{1+\mu}(x), \quad \forall x \neq 0,
\]
where \( \mu \) is the homogeneity degree of the vector field \( f \) and \( q > 0 \) is a positive parameter.

In other words, any stable homogeneous system admits a homogeneous Lyapunov function. Notice that, for \( \mu < 0 \) the inequality (15) repeats the inequality (2). The estimate (15) yields the following corollary which characterize the convergence rates of homogeneous systems.

Corollary 1. [22] If the system is asymptotically stable then it is

- globally uniformly finite-time stable\(^3\) for \( \mu < 0 \);
- globally uniformly exponentially stable for \( \mu = 0 \);
- globally uniformly nearly fixed-time stable\(^4\) for \( \mu > 0 \),

where \( \mu \in \mathbb{R} \) is a homogeneity degree of the vector field \( f \).

\(^3\)A system is globally uniformly finite-time stable if it is Lyapunov stable and there exists a locally bounded function \( T : \mathbb{R}^n \rightarrow [0, +\infty) \) such that \( \|x(t)\| = 0, \forall t \geq T(x_0), \forall x_0 \in \mathbb{R}^n \). For more details about finite-time stability see [5], [32].

\(^4\)A system is globally uniformly nearly fixed-time stable if it Lyapunov stable and \( \forall r > 0, \exists T_r > 0 : \|x(t)\| < r, \forall t \geq T_r \) independently of \( x_0 \in \mathbb{R}^n \). For more details about fixed-time stability see [24].
One of useful properties of homogeneous control systems is robustness (Input-to-State Stability) in the sense of the following definition.

**Definition 5.** [34] The system

$$\dot{x} = f(x, \delta), \quad f : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n}, \quad f \in C(\mathbb{R}^{n+k})$$

(16)

is said to be Input-to-State Stable (ISS) if there exist\(^5\) $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$:

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma \left( \sup_{\tau \in [0,t]} \|\delta(\tau)\| \right), \quad \forall t > 0.$$  

The ISS analysis is a non-trivial problem in the general case [35], however, ISS of a perturbed homogeneous systems follows from asymptotic stability of a non-perturbed one provided that the disturbances are involved to the system in a homogeneous way [33], [12], [1].

**Theorem 4.** [3] Let $d$ be a dilation in $\mathbb{R}^{n}$, $\tilde{d}$ is a dilation in $\mathbb{R}^{k}$, and $t \mu \in \mathbb{R}$. If

$$f(d(s)x, \tilde{d}(s)\delta) = e^{\mu s}d(s)f(x, \delta), \quad \forall s \in \mathbb{R}, \forall x \in \mathbb{R}^{n}, \forall \delta \in \mathbb{R}^{k}.$$  

and the system (16) with $\delta = 0$ is asymptotically stable then this system is ISS.

**Example 3.** The control system considered in Example 2 is robust (Input-to-State Stable) with respect to bounded additive exogenous perturbations and additive measurement noises. Indeed, let us denote

$$f(x, \delta) = \left( \begin{array}{c} x_2 \\ u(x + \delta) \end{array} \right) + \delta_1, \quad x = (x_1, x_2)^{\top} \in \mathbb{R}^2, \quad \left( \begin{array}{c} \delta_1 \\ \delta_2 \end{array} \right) \in \mathbb{R}^4,$$

where the feedback control $u$ is defined in Example 2, $\delta_1 \in \mathbb{R}^2$ is a measurement noise and $\delta_2$ is an exogenous perturbation. Let the dilation $d$ be defined as in Example 2. Then

$$\tilde{d}(s) = \left( \begin{array}{cc} d(s) & 0 \\ 0 & e^{-2s}d(s) \end{array} \right),$$

is a dilation in $\mathbb{R}^4$ with the generator

\[
G_{\tilde{d}} = \begin{pmatrix}
5 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

---

\(^5\)A function $\gamma : [0, +\infty) \rightarrow [0, +\infty)$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$.

A function $\beta : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is of class $\mathcal{KL}$ if the function $r \rightarrow \beta(r, s)$ is of class $\mathcal{K}$ for any fixed $s \in \mathbb{R}$ and for any fixed $r \geq 0$ the function $s \rightarrow \beta(r, s)$ is decreasing to zero as $s \rightarrow +\infty$. 


being an anti-Hurwitz matrix. In Example 2, it was shown that the system
\[ \dot{x} = f(x, 0) \]
is globally uniformly asymptotically stable. Since
\[ f(d(s)x, \tilde{d}(s)\delta) = e^{-2s}d(s)f(x, \delta), \quad \forall x \in \mathbb{R}^2, \quad \forall \delta \in \mathbb{R}^4, \forall s \in \mathbb{R} \]
then, by Theorem 4, this system is ISS.

4. Homogeneous controllability functions for linear plants

Let us consider the linear control system
\[ \dot{x} = Ax + Bu, \quad t > 0, \quad x(0) = x_0 \] (17)
where \( x(t) \in \mathbb{R}^n \) is the system state, \( A \in \mathbb{R}^{n \times n} \) is the system matrix, \( u(t) \in \mathbb{R}^m \) is a control input and \( B \in \mathbb{R}^{n \times m} \).

Inspired by Theorem 1 and [26] we present the following result.

Theorem 5. Let the pair \( \{A, B\} \) be controllable and the pair \( Y_0 \in \mathbb{R}^{m \times n}, G_0 \in \mathbb{R}^{n \times n} \) be a solution of the linear algebraic equation
\[ AG_0 - G_0A + BY_0 = A, \quad G_0B = 0 \] (18)
with respect to \( G_0, Y_0 \), then

1) \( G_0 = I_n + \mu G_0 \) is anti-Hurwitz for any \( \mu \in [-1, 0) \);
2) \( G_0 - I_n \) is invertible and the matrix \( A_0 = A + BY_0(G_0 - I_n)^{-1} \) satisfies
\[ A_0G_d = (G_d + \mu I_4)A_0, \quad G_dB = B. \] (19)

If a solution \( X \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{m \times n} \) satisfies the system of linear matrix inequalities (LMIs)
\[ A_0X + XA_0^\top + BY + Y^\top B^\top \prec 0, \quad G_dX + XG_d^\top \succ 0, \quad X = X^\top \succ 0 \] (20)
then

3) the canonical homogeneous \( \| \cdot \|_d \) induced by the weighted Euclidean norm
\[ \| x \| = \sqrt{x^\top X^{-1}x} \]
is a controllability function of the system (17) with the control
\[ u(x) = Y_0(G_0 - I_n)^{-1}x + \| x \|^\mu X^{-1}d(- \ln \| x \|_d)x \] (21)
and
\[ \frac{d}{dt}\| x \|_d \leq \frac{\lambda_{\max}(A_0X + XA_0^\top + BY + Y^\top B^\top)}{\lambda_{\max}(G_dX + XG_d^\top)} \| x \|_d^{1 + \mu}, \quad x \neq 0, \] (22)
where \( \lambda_{\max}(Q) \) denotes a maximal eigenvalue of a symmetric matrix \( Q \) and \( d \) is a dilation generated by \( G_d \).
4) \( u \in C^1(\mathbb{R}^n\setminus\{0\}) \) for any \( \mu \in [-1, 0) \), \( u \) is continuous at zero if \( \mu \in (-1, 0) \) and \( u \) is discontinuous at zero but locally bounded if \( \mu = -1 \); 

5) the closed-loop system (17), (21) is \( \mathbf{d} \)-homogeneous of the degree \( \mu \).

Proof. 1) Since the pair \( \{A, B\} \) then there exists [21], [28] a coordinate transformation \( M \in \mathbb{R}^{n \times n} \) such that 

\[
M^{-1}B = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ H_{k,k+1} & \cdots & 0 \end{pmatrix}, \quad M^{-1}A_0M = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k1} & A_{k2} & A_{k3} & \cdots & A_{kk} \end{pmatrix},
\]

(23)

where \( A_{i,j} \in \mathbb{R}^{n_i \times n_j}, \) \( \text{rank}(A_{i,j}) = n_i, n_i \leq n_{i+1}, i = 1, 2, \ldots, k, n_1 + \cdots + n_k = n, n_k \leq m, n_{k+1} = m \) and \( k \) is a minimal natural number such that \( \text{rank}(B, AB, \ldots, A^{k-1}B) = n \).

In this case, the equation (18) can be equivalently rewritten as follows

\[
\ddot{A}G_0 - \ddot{G}_0 \dddot{A} + B\dot{Y}_0 = \dddot{A}, \quad \ddot{G}_0 \dddot{B} = 0
\]

(24)

where \( \dddot{B} = M^{-1}B, \dddot{A} = M^{-1}A_0M, \dddot{G}_0 = M^{-1}G_0M \) and \( \dot{Y}_0 = Y_0M \).

Taking into account, the structure of \( \dddot{A} \) and \( \dddot{B} \) we conclude that the linear equation (24) has a solution with respect to \( \dot{Y}_0, \dddot{G}_0 \) and the matrix \( \dddot{G}_0 \) has the block lower triangular form

\[
\dddot{G}_0 = \begin{pmatrix}
-(k-1)I_{n_1} & 0 & 0 & \cdots & 0 \\
* & -(k-2)I_{n_2} & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \vdots \\
& * & * & * & -I_{n_{k-1}} \\
& * & * & * & * & 0
\end{pmatrix},
\]

where \( * \) denotes a possibly non-zero block. The latter means \( G_0 - I \) is invertible and \( G_0 = I_n + \mu G_0 \) is anti-Hurwitz for \( \mu < 0 \).

2) Since \( G_0 - I_n \) is invertible and the pair \( G_0, Y_0 \) satisfies (18) then

\[
A - A + \mu AG_0 - \mu G_0 A + \mu BY_0 = \mu A
\]

or, equivalently,

\[
AG_d + \mu BY_0 = (G_d + \mu I_n)A.
\]

On the one hand, obviously, \( G_d B = B \) and

\[
(G_d + \mu I_n)A = (G_d + \mu I_n) \left( A + BY_0(G_0 - I_n)^{-1} - BY_0(G_0 - I_n)^{-1} \right) =
\]

\[
(G_d + \mu I_n)A - (1 + \mu)BY_0(G_0 - I_n)^{-1}.
\]

On the other hand, one has

\[
AG_d + \mu BY_0 = \left( A + BY_0(G_0 - I_n)^{-1} - BY_0(G_0 - I_n)^{-1} \right) G_d + \mu BY_0 =
\]
\[ A_0 G_d - B Y_0 (G_0 - I_n)^{-1} G_d + \mu B Y_0 = A_0 G_d - B Y_0 (G_0 - I_n)^{-1} (G_d - \mu (G_0 - I_n)) = A_0 G_d - (\mu + 1) B Y_0 (G_0 - I_n)^{-1}. \]

Hence, we derive the identity (19).

3) First of all, notice that, the canonical homogeneous is well defined since \( X^{-1} \) is satisfies the second and the third inequalities from (20). It is continuous on \( \mathbb{R}^n \) and continuously differentiable on \( \mathbb{R}^n \setminus \{0\} \) in the view of Lemma 2.

On the other hand, the formula \( d(s) = e^{sG_d} \) and the identity (19) yield

\[ A_0 d(s) = \sum_{i=0}^{\infty} \frac{s^i A_0 G_d}{i!} = \sum_{i=0}^{\infty} \frac{s^i (G_d - \mu I_n)^i A_0}{i!} = e^{sG_d} A_0 = e^{s u} d(s) A_0. \] (25)

and

\[ d(s) B = e^{s} B, \quad \forall s \in \mathbb{R}. \] (26)

In this case, for the closed loop system, using the formula (12) we derive

\[
\begin{align*}
\frac{d}{dt} ||x||_d &= ||x||_d x^\top d^\top (-\ln ||x||_d) X^{-1} (-\ln ||x||_d) d(-\ln ||x||_d) x \\
&= x^\top d^\top (-\ln ||x||_d) X^{-1} G_d (-\ln ||x||_d) x \\
&= x^\top d^\top (-\ln ||x||_d) X^{-1} G_d (-\ln ||x||_d) x \\
&= x^\top d^\top (-\ln ||x||_d) X^{-1} (A_0 Y + X A_0 Y + X Y B^\top) X^{-1} d(-\ln ||x||_d) x,
\end{align*}
\]

Hence, taking into account that \( X \) and \( Y \) satisfy (20) we derive (22).

4) Since the canonical homogeneous is continuous on \( \mathbb{R}^n \) and continuously differentiable on \( \mathbb{R}^n \setminus \{0\} \) then, by construction, \( u \) is continuously differentiable on \( \mathbb{R}^n \setminus \{0\} \). Moreover, from the definition of the canonical homogeneous norm we have

\[ x^\top d^\top (-\ln ||x||_d) X^{-1} d(-\ln ||x||_d) x = 1, \]

so

\[ ||u|| \leq r_1 ||x|| + r_2 ||x||^\mu_1, \quad \forall x \in \mathbb{R}^n \]

with \( r_1 \geq 0 \) and \( r_2 > 0 \) dependent of \( Y_0, G_0, X, \) and \( Y \). The latter means that \( u \) is locally bounded for any \( \mu \in [-1, 0) \) and continuous at zero for \( \mu \in (-1, 0) \).

5) Let us denote the right-hand side of the closed-loop system by

\[ f(x) = A_0 x + ||x| |^\mu_1 B Y X^{-1} d(-\ln ||x||_d) x \]

Using the identities (25), (26) and \( ||d(s)||_d = e^{s/||x||_d} \) we derive

\[
\begin{align*}
f(d(s)x) &= A_0 d(s)x + ||d(s)||_d^\mu_1 B Y X^{-1} d(-\ln ||d(s)||_d) x \\
&= e^{s} d(s)x + e^{s(1+\mu)/||x||_d} B Y X^{-1} d(-\ln ||x||_d x) \\
&= e^{s} d(s) A_0 x + e^{s(1+\mu)/||x||_d} B Y X^{-1} d(-\ln ||x||_d) x \\
&= e^{s} d(s) \left( A_0 x + ||x||^\mu_1 B Y X^{-1} d(-\ln ||x||_d) x \right) = e^{s} d(s) f(x),
\end{align*}
\]

for all \( x \in \mathbb{R}^n \) and all \( s \in \mathbb{R} \). The proof is complete.
Remark 1. A solution of the linear matrix equation (18) is not unique in the general case of the multiply-input linear control system. In the single input case \( m = 1 \), the solution is always unique and \( \tilde{G}_0 \) is a diagonal matrix.

Remark 2. The system of linear matrix inequalities (20) is always feasible at least for \( \mu \) close to 0. Indeed, the feasibility of the first LMI together with the third one follows from controllability of the pair \( \{A, B\} \) (see, [6]), while the second LMI becomes the third one for \( \mu \) tending to 0 (reps. \( G_d \to I_n \) as \( \mu \to 0 \)). Notice that, the system of LMIs (20) is always feasible for \( \mu = [-1,0) \), at least, in the single input case [25].

As a straightforward corollary of Theorems 4 and 5 (see also Example 3 as the hint of the proof) we provide the following result.

Corollary 2. The control system (17), (21) designed by Theorem 5 is ISS with respect bounded additive measurement noises if \( \mu \in [-1,0) \) and ISS with respect to bounded additive exogenous perturbations if \( \mu \in (-1,0) \).

Remark 3. If \( m = 1 \) and \( A \) as in Example 1 then \( Y_0 = 0 \),

\[
G_0 = \begin{pmatrix}
-n+1 & 0 & \cdots & 0 & 0 \\
0 & -n+2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -1 & 0 \\
0 & 0 & \cdots & 0 & -1+n \\
-1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad d(s) = \begin{pmatrix}
e^{s(1-\mu(n-1))} & 0 & \cdots & 0 & 0 \\
0 & e^{1-\mu(n-2)} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & e^{(1-\mu)} & 0 \\
0 & 0 & \cdots & 0 & e^s
\end{pmatrix}.
\]

Obviously, for \( F = a_0 X^{-1} \), an integer \( \alpha = -\frac{1}{\mu} \) and \( m = \alpha + n - 1 \) the canonical homogeneous norm defined in Theorem 5 simply coincides with the controllability function given by (4), (6) and the system of LMIs (20) implies (5), (7) with \( K = YX^{-1} \). Therefore, the controller (21) coincides with (8). The latter means that the control system (3), (8) is \( d \)-homogeneous.

Similar conclusions can be made for controllability functions designed in [17, Chapter 1, §5] and [17, Chapter 1, §6].

5. Conclusions

The paper shown that linear autonomous control system always admits a generalized homogeneous controllability function. Moreover, the well-known controllability function studied in [17, Chapter 1] are homogeneous as well! This immediately results in robustness (Input-to-State Stability) of the corresponding control systems with respect to a rather large class of disturbances. An interesting open problem in this context is the homogeneity analysis of controllability functions designed by means of integral operators (see [17, Chapter 5]).

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On homogeneous controllability functions
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The controllability function method, introduced by V. I. Korobov in late 1970s, is known to be an efficient tool for control systems design. It is developed for both linear/nonlinear and finite/infinite dimensional systems. This paper bridges the method with the homogeneity theory popular today. The standard homogeneity known since 18th century is a symmetry of function with respect to uniform scaling of its argument. Some generalizations of the standard homogeneity were introduced in 20th century. This paper shows that the so-called homogeneous norm is a controllability function of the linear autonomous control system and the corresponding closed-loop system is homogeneous in the generalized sense. This immediately yields many useful properties known for homogeneous systems such as robustness (Input-to-State Stability) with respect to a rather large class of perturbations, in particular, with respect to bounded additive measurement noises and bounded additive exogenous disturbances. The main theorem presented in this paper slightly refines the design of the controllability function for a multiply-input linear autonomous control systems. The design procedure consists in solving subsequently a linear algebraic equation and a system of linear matrix inequalities. The homogeneity itself and the use of the canonical homogeneous norm essentially simplify the design of a controllability function and the analysis of the closed-loop system. Theoretical results are supported with examples. The further study of homogeneity-based design of controllability functions seems to be a promising direction for future research.

Keywords: controllability function; generalized homogeneity; robustness.

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