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Non-smooth systems of generalized MIMO triangular form

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Following work [Korobov V.I., Pavlichkov S.S. Global properties of the triangular systems in the singular case// J. Math. Anal. Appl. – 2008. – **342**. – P. 1426-1439], we consider the class of MIMO (multi-input and multi-output) triangular systems of the so-called "generalized triangular form"defined by the same conditions as in [Korobov V.I., Pavlichkov S.S. Global properties of the triangular systems in the singular case // J. Math. Anal. Appl. – 2008. – **342**. – P. 1426-1439] except the following one: in comparison with the above-mentioned work, we remove the assumption about C^1 - smoothness of the dynamics and assume that the dynamics satisfies the local Lipschitz condition w.r.t. states and controls only. In the current work, is proven that such a new class is globally controllable (by means of controls from class C^μ with an arbitrary $\mu \geq 0$ including C^∞). The background of the proof is a modification of the construction proposed in [Korobov V.I., Pavlichkov S.S. Global properties of the triangular systems in the singular case // J. Math. Anal. Appl. – 2008. – **342**. – P. 1426-1439]. Whereas this new class is wider than that from [Korobov V.I., Pavlichkov S.S. Global properties of the triangular systems in the singular case // J. Math. Anal. Appl. – 2008. – **342**. – P. 1426-1439] to some extent, we prove its global controllability only (while in the above-mentioned work stronger results were obtained for the C^1 - case).

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1. Introduction.

This work is devoted to nonlinear systems of the form

$$\begin{cases} \dot{x}_i = f_i(x_1, ..., x_{i+1}), \quad i = 1, ..., n-1; \\ \dot{x}_n = f_n(x_1, ..., x_n, u); \end{cases} \tag{1}$$

which is called "triangular"form (TF). Beginning with Korobov's work [10], this class has been explored extensively in various directions - [1, 3, 6, 11, 12, 13, 17, 18, 20]. This interest was motivated by different strands of research in nonlinear control theory: exact linearization and feedack equivalence [6, 7], stabilization and adaptive control [8, 14, 20], mechanics and other applications - [13].

Originally, TF was considered in the so-called "regular case" $\frac{\partial f_i}{\partial x_{i+1}} \neq 0$, $1 \leq i \leq n$, which means [7, 10] that (1) can be brought to the Brunovsky linear canonical form by means of a diffeomorphism $z = \Phi(x)$, $v = \Psi(x, u)$ at least locally - see the well-known Jakubczyk-Respondek-Hunt-Su-Meyer-Krischenko conditions. Although (to our best knowledge) the first work devoted to the singular case was [17] (1986), during the last 15 years, the singular case has received a lot of attention - [3, 21, 12]. In the latter work [12], the authors tried to intord루ce and study as wide class of the TF as possible so that this new class could be potentially treated as a global nonlinear analog of the Brunovsky forms $\dot{x}_1 = x_2, \ldots, \dot{x}_{n-1} = x_n$, $\dot{x}_n = u$. This naturally leads to the class defined by the following conditions 1 and 2 only - [12]:

1. *The dynamics of the system (1) is continuously differentiable i.e.,* $f_i \in C^1$

(It is natural to require some smoothness at least of class C^1, if we want to speak of this generalized TF in the context of feedback equivalence, which is apparently possible - see [15])

2. *All the functions* $f_i(t, x_1, ..., x_i, \cdot)$, $i = 1, \ldots, n$, *are surjective*

(It is necessary ro require some dependence of each f_i on x_{i+1}, because, if some f_i does not depend on x_{i+1} at all, the triangular system has an uncontrollable part. Condition 2 is a global nonlinear analog of the local condition $\frac{\partial f_i}{\partial x_{i+1}} \neq 0$, $1 \leq i \leq n$, and condition 2 seems to be as general as possible if one wants to speak of global controllability)

Note that system (1) can be MIMO, i.e., x_i and u are not necessarily scalar in conditions 1, 2.

However, it is natural to try to generalize assumption 1 to some extent: instead of C^1 - smoothness, one may want to require the local Lipschitz condition only. In this case, first, the construction from [12] should be revised significantly (because the assumption $f_i \in C^1$ was essential in [12]), and, second, some properties obtained in [12, 15] are lost (it is impossible to speak of feedback equivalence and it is not clear how to obtain main Theorems 3.1, 3.2 from [12]).

Nevertheless, it appears that systems of such a new class are globally controllable at least. The goal of the current paper is to prove the global controllability for this new class.

2. Main result.

We consider a control system of the following form

$$\begin{cases} \dot{x}_i = f_i(t, x_1, ..., z_{i+1}), \quad i = 1, ..., \nu - 1; \\ \dot{x}_\nu = f_\nu(t, x_1, ..., x_\nu, u); \end{cases} \qquad t \in I = [t_0, T] \tag{2}$$

where $x{=}(x_1,...,x_\nu)^T{\in}\mathbf{R}^n = \mathbf{R}^{m_1+\ldots+m_\nu}$ is the state (with $x_i{\in}\mathbf{R}^{m_i}$ and $n{=}m_1{+}...{+}m_\nu$), $u{\in}\mathbf{R}^{m_{\nu+1}}$ is the control. We assume that the following conditions hold

(i) $f_i \in C(I \times \mathbf{R}^{m_1} \times \ldots \times \mathbf{R}^{m_{i+1}}; \mathbf{R}^{m_i})$, $i = 1, \ldots, \nu$;

(ii) *Each f_i satisfies the local Lipschitz condition w.r.t. (x,u), i.e., for every $i = 1, \ldots, \nu$ and every compact set $K \subset \mathbf{R}^{m_1} \times \ldots \times \mathbf{R}^{m_{i+1}}$ there exists $L_K > 0$ such that , for each $t{\in}I$, each $(x_1^1,...,x_{i+1}^1)^T{\in}K$ and each $(x_1^2,...,x_{i+1}^2)^T{\in}K$, we have:*

$$|f_i(t,x_1^1,...,x_{i+1}^1) - f_i(t,x_1^2,...,x_{i+1}^2)| \le L_K \sum_{j=1}^{i+1} |x_j^1 - x_j^2|.$$

(iii) *For every $i{=}1,...,\nu$, and every $(t,x_1,...,x_i) \in I \times \mathbf{R}^{m_1+\ldots+m_i}$, we have $f_i(t,x_1,...,x_i,\mathbf{R}^{m_{i+1}}) = \mathbf{R}^{m_i}$.*

Our main result is as follows.

Theorem 1 *Assume that system (2) satisfies conditions (i),(ii),(iii). Then, for every $\mu{\in}\mathbf{Z}_+{\cup}\{\infty\}$, system (2) is globally controllable by means of controls from class $C^\mu(I;\mathbf{R}^{m_{\nu+1}})$.*

Throughout the paper, for each $\tau{\in}I$, each $x_0{\in}\mathbf{R}^n$, and each $u(\cdot){\in}L_\infty(I;\mathbf{R}^{m_{\nu+1}})$, by $t{\mapsto}x(t,\tau,x^0u(\cdot))$ we denote the trajectory, of system (2), defined by this control $u(\cdot)$ and by the initial condition $x(\tau){=}x^0$ on some maximal subinterval $J{\subset}I$. (Thus, by the well-known definition, Theorem 1 states that, whatever $\mu{\in}\mathbf{Z}_+{\subset}\{\infty\}$, for each $x^0{\in}\mathbf{R}^n$ and each $x^T{\in}\mathbf{R}^n$ there is $u(\cdot)$ in $C^\mu(I;\mathbf{R}^{m_{\nu+1}})$ such that $x(t,t_0,x^0,u(\cdot))$ is defined for all $t{\in}I$ and $x(T,t_0,x^0,u(\cdot)) = x^T$).

2. Proof of Theorem 1

The background of the proof is a certain backstepping technique which was developed in our previous related works devoted to the triangular forms which are not feedback linearizable - [11, 12, 15]. Roughly speaking, this means that we resolve the problem of controllability by induction over $\nu = 1, 2, \ldots$. The main obstacle we should obviate when following this pattern is that the dynamics of our system (2) is no longer of class C^1 (instead we have assumptions (i)-(ii) only). On the other hand, in all the above-mentioned works, the assumption on C^1 - smoothness was essential as this argument begins with studying the linearized control systems around a certain trajectory. To handle this problem, we note that, in [12], the controllability of the linearized control system was used locally in some neighborhood of some regular point only. The construction beyond this neighborhood does not refer to any C^1 - smoothness at all (see [12], Section 6). Thus, it is natural to try to modify the argument from [12] accordingly. As we can see from the proof, the disadvantage of such an extension is that, in contrast to [12], we cannot speak of families of controls resolving the controllability problem and, in particular, we cannot prove the controllability of the uniformly bounded perturbations of system (2) (this is again because some properties of the linearized control system are essential, when construction such families in [12]). However such a modification does work when proving global controllability (at least).

Fix an arbitrary p in $\{1,\dots,\nu\}$. Define $k{:=}m_1{+}...{+}m_p$, $y{:=}(x_1,...,x_p)^T$ (with $x_i{\in}\mathbf{R}^{m_i}$, $i{=}1,...,p$) and consider the k - dimensional control system

$$\begin{cases} \dot{x}_i = f_i(t,x_1,...,x_{i+1}), \quad i=1,...,p-1; \\ \dot{x}_p = f_p(t,x_1,...,x_p,v); \end{cases} \qquad t\in I \tag{3}$$

with states $y=(x_1,...,x_p)^T\in\mathbf{R}^k=\mathbf{R}^{m_1+\dots+m_p}$ and controls $v\in\mathbf{R}^{m_{p+1}}$.

Given $y\in\mathbf{R}^k$, $\tau\in I$, and $v(\cdot)$ in $L_\infty(I;\mathbf{R}^{m_{p+1}})$, by $t\mapsto y(t,\tau,y,v(\cdot))$ denote the trajectory, of system (3), defined by the control $v(\cdot)$ and by the initial condition $y(\tau,\tau,y,v(\cdot))=y$ on some maximal subinterval $J\subset I$.

The proof of Theorem 1 is in two steps.

Step 1. Let us first reduce Theorem 1 to the following statement.

Theorem 2 *Let p be in $\{1,...,\nu\}$. Assume that, for every $y^0\in\mathbf{R}^k$ and every $\delta>0$, there exists a family of functions $\{y(\eta,\cdot)=(x_1(\eta,\cdot),...,x_p(\eta,\cdot))\}_{\eta\in\mathbf{R}^k}$ such that:*

1) The map $\eta\mapsto y(\eta,\cdot)$ is of class $C(\mathbf{R}^k;C^1(I;\mathbf{R}^k))$

2) For each $\eta\in\mathbf{R}^k$ we have

$$\dot{x}_i(\eta,t)=f_i(t,x_1(\eta,t),\dots,x_{i+1}(\eta,t)),\quad t\in I,\quad 1\le i\le p-1;$$

(if $p=1$, then we have 0 equalities, which means that Condition 2) is omitted by definition)

3) $y(\eta,t_0)=y^0$ and $|y(\eta,T)-\eta|<\delta$ for all $\eta\in\mathbf{R}^k$

Then, for every $(y^0,y^0_{k+1})\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}$, every $\varepsilon>0$ and every $\mu\in\mathbf{Z}_+\cup\{\infty\}$, there exists a family of controls $\{\hat{v}_{(\eta,\beta)}(\cdot)\}_{(\eta,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}}$ such that

4) The map $(\eta,\beta)\mapsto\hat{v}_{(\eta,\beta)}(\cdot)$ is of class $C(\mathbf{R}^k\times\mathbf{R}^{m_{p+1}};C^\mu(I;\mathbf{R}^{m_{p+1}}))$

5) For each $(\eta,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}$, we have $\hat{v}_{(\eta,\beta)}(T)=\beta$ and $\hat{v}_{(\eta,\beta)}(t_0)=y^0_{k+1}$.

6) For every $(\eta,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}$, the trajectory $t\mapsto y(t,t_0,y^0,\hat{v}_{(\eta,\beta)}(\cdot))$ is well-defined for all $t\in I$ and $|y(T,t_0,y^0,\hat{v}_{(\eta,\beta)}(\cdot))-\eta|<\varepsilon$.

Let us first show that Theorem 2 implies Theorem 1. Assume that Theorem 2 is proved.

For $p=1$, consider the corresponding m_1 - dimensional system (3) and pick any $y^0_1\in\mathbf{R}^{m_1}$. Then, for each $\delta>0$, we easily get the existence of a family $\{y(\xi,\cdot)\}_{\xi\in\mathbf{R}^{m_1}}=\{x_1(\xi,\cdot)\}_{\xi\in\mathbf{R}^{m_1}}$ such that conditions 1)-3) of Theorem 2 hold (for instance, the family $x_1(\xi,t):=y^0_1+\frac{t-t_0}{T-t_0}(\xi-y^0_1)$ will suit). Then, for $p{=}1$, we obtain: for every $\varepsilon{>}0$ every $(y^0_1,y^0_2){\in}\mathbf{R}^{m_1+m_2}$ and every $\mu\ge1$, there is a family of controls $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbf{R}^{m_1}\times\mathbf{R}^{m_2}}$ such that conditions 4), 5), 6) of Theorem 2 hold with $p{=}1$.

Suppose $p{=}2$. Given any $\delta{>}0$, and any $y^0{=}(y^0_1,y^0_2){\in}\mathbf{R}^{m_1+m_2}$, put $\varepsilon{:=}\delta$, and (for this $\varepsilon>0$) find the family $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbf{R}^{m_1}\times\mathbf{R}^{m_2}}$ obtained at the previous step (with $p = 1$). Then conditions 4)-6) for $p = 1$ imply that the family $\{y(\eta, \cdot)\}_{\eta=(\xi,\beta)\in\mathbf{R}^{m_1}\times\mathbf{R}^{m_2}}$ given by

$$y(\xi, \beta, t) := (y(t, t_0, y_1^0, \hat{v}_{(\xi,\beta)}(\cdot)), \hat{v}_{(\xi,\beta)}(t)), \qquad t \in I, \qquad \eta = (\xi, \beta) \in \mathbf{R}^{m_1}\times\mathbf{R}^{m_2}$$

satisfies all the conditions 1), 2), 3) of Theorem 2 with $p = 2$. Then we can apply Theorem 2 to $p = 2$, etc.

Arguing similarly by induction over $p = 1, \dots, \nu$, we obtain (for $p = \nu$) that for each $\varepsilon > 0$, each $\mu \in \mathbf{N} \cup \{\infty\}$, each $x^0 \in \mathbf{R}^\nu$, and each $\alpha = y^0_{\nu+1}$ there exists a family of controls $\{\hat{v}_{(\eta,\beta)}(\cdot)\}_{(\eta,\beta)\in\mathbf{R}^n\times\mathbf{R}^{m_{\nu+1}}}$ such that conditions 4), 5), 6) of Theorem 2 hold for $p = \nu$. Fix any $\beta \in \mathbf{R}^{m_{\nu+1}}$ and define the family of controls $\{u_\eta(\cdot)\}_{\eta\in\mathbf{R}^n}$ by $u_\eta(t) := \hat{v}_{(\eta,\beta)}(t)$ for all $\eta \in \mathbf{R}^n$. Then $\{u_\eta(\cdot)\}_{\eta\in\mathbf{R}^n}$ satisfies the following conditions:

(a) $\eta \mapsto u_\eta(\cdot)$ is of class $C(\mathbf{R}^n; C^\mu(I; \mathbf{R}^{m_{\nu+1}}))$

(b) For each $\eta \in \mathbf{R}^n$, the trajectory $t \mapsto x(t, t_0, x^0, u_\eta(\cdot))$ is well-defined and $|x(T, t_0, x^0, u_\eta(\cdot)) - \eta| < \varepsilon$.

Given any $\mu \in \mathbf{N}\bigcup\{\infty\}$ any $\varepsilon > 0$ an arbitrary $x^0 \in \mathbf{R}^n$, and an arbitrary $x^T \in \mathbf{R}^n$, let $\{u_\eta(\cdot)\}_{\eta\in\mathbf{R}^n}$ be a family of controls such that (a), (b) hold. Let us prove the existence of $\eta^*\in\mathbf{R}^n$ such that $x(T, t_0, x^0, u_{\eta^*}(\cdot)) = x^T$. Indeed, by condition (a), the map $\eta \mapsto \eta - x(T, t_0, x^0, u_\eta(\cdot)) + x^T$ is of class $C(\mathbf{R}^n; \mathbf{R}^n)$ (condition (b) implies that this map is well-defined). On the other hand, from condition (b), it follows that the image of this map belongs to $\overline{B_\varepsilon(x^T)}$. In particular, $\eta \mapsto \eta - x(T, t_0, x^0, u_\eta(\cdot)) + x^T$ is a continuous map of a compact, convex set $\overline{B_\varepsilon(x^T)}$ into $\overline{B_\varepsilon(x^T)}$, and then, using the Brouwer fixed-point theorem, we get the existence of $\eta^* \in \mathbf{R}^n$ such that

$$\eta^* = \eta^* - x(T, t_0, x^0, u_{\eta^*}(\cdot)) + x^T, \qquad \text{i.e.,} \qquad x(T, t_0, x^0, u_{\eta^*}(\cdot)) = x^T.$$

Thus, for every $x^0 \in \mathbf{R}^n$, and every $x^T \in \mathbf{R}^n$, there is a control $u_{\eta^*}(\cdot) \in C^\mu(I; \mathbf{R}^{m_{\nu+1}})$ such that $x^T = x(T, t_0, x^0, u_{\eta^*}(\cdot))$.

We have proved that Theorem 2 implies Theorem 1. Thus, it suffices to prove Theorem 2.

Step 2: Proof of Theorem 2. Fix an arbitrary p in $\{1, \dots, \nu\}$ arbitrary $(y^0, y^0_{k+1}) \in \mathbf{R}^k \times \mathbf{R}^{m_{p+1}}$, arbitrary $\varepsilon > 0$, and $\mu \in \mathbf{Z}_+ \cup \{\infty\}$. Define $\delta := \frac{\varepsilon}{4}$ and let $\{y(\eta, \cdot)\}_{\eta\in\mathbf{R}^k}$ satisfy conditions 1)-3) of Theorem 2.

To prove Theorem 2, we modify the pattern proposed in [12]. Along with (3), we consider the k - dimensional system

$$\begin{cases} \dot{x}_i(t) = f_i(t, x_1(t), ..., x_{i+1}(t)), & i = 1, ..., p-1; \\ \dot{x}_p(t) = \omega(t); \end{cases} \qquad t \in I \tag{4}$$

where $x(t) = (x_1(t), ..., x_p(t))^T \in \mathbf{R}^k$ is the state and $\omega(t) \in \mathbf{R}^{m_p}$ is the control. For every $y \in \mathbf{R}^k$, every $\tau \in I$, and every $\omega(\cdot) \in L_\infty(I; \mathbf{R}^{m_p})$, let $t \mapsto z(t, \tau, y, \omega(\cdot))$

denote the trajectory, of system (4), defined by this control $\omega(\cdot)$ and by the initial condition $z(\tau,\tau,y,\omega(\cdot)) = y$ on some maximal subinterval $J \subset I$. Using conditions 2), 3) of Theorem 2, we obtain:

$$y(\eta,t) = z(t,t_0,y^0,\dot{x}_p(\eta,\cdot)) \qquad \text{for all} \quad \eta \in \mathbf{R}^k, \ \ t \in I \tag{5}$$

Then we apply the Gronwall-Bellmann lemma and (arguing as in [16] Lemma 4.3, page 97) obtain the existence of $\sigma(\cdot)$ in $C(\mathbf{R}^k;]0,+\infty[)$ such that, for every $\eta \in \mathbf{R}^k$ and every $\omega(\cdot) \in L_\infty(I;\mathbf{R}^{m_{p+1}})$, we have:

$$\forall\, t \in I \quad |z(t,t_0,y^0,\omega(\cdot)) - y(\eta,t)| < \delta,$$

$$\text{whenever} \quad \|\, \omega(\cdot) - \dot{x}_p(\eta,\cdot)\|_{L_\infty(I;\mathbf{R}^{m_{p+1}})} < \sigma(\eta) \tag{6}$$

Next, arguing as in the proof of Lemma 5.1 from [12], we construct a family $\{v(\eta,\cdot)\}_{\eta\in\mathbf{R}^k}$ of controls defined on I and a function $M(\cdot)$ in $C(\mathbf{R}^k;]0,+\infty[)$ such that:

(a$_1$) For each $\eta \in \mathbf{R}^k$, the control $v(\eta,\cdot)$ is a piecewise constant function on I and the map $\eta \mapsto v(\eta,\cdot)$ is of class $C(\mathbf{R}^k; L_1(I;\mathbf{R}^{m_{p+1}}))$.

(a$_2$) For every $\eta \in \mathbf{R}^k$, the trajectory $t \mapsto y(t,t_0,y^0,v(\eta,\cdot))$ is defined for all $t\in I$ and

$$|\dot{x}_p(\eta,\cdot) - f_p(t, y(t,t_0,y^0,v(\eta,\cdot)), v(\eta,t))| < \sigma(\eta) \quad \text{for all} \ \ t \in I, \ \ \eta \in \mathbf{R}^k$$

(a$_3$) For every $\eta \in \mathbf{R}^k$, we have $\|\ v(\eta,\cdot)\ \|_{L_\infty(I;\mathbf{R}^k)} < M(\eta)$

Remark. Note that the proof of Lemma 5.1 from [12] does not actually require any smoothness of f_i. In [12], the condition $f_i \in C^1$ is used only when studying the linearized control system in some neighborhood of some regular point. When proving Lemma 5.1 from [12], we need only the existence and uniqueness of the solution of the corresponding Cauchy problem. This is guaranteed by assumptions (i), (ii) from the current paper as well, and, therefore, it is possible to repeat this argument and get the existence of $\{v(\eta,\cdot)\}$ and $M(\cdot)$ satisfying (a$_1$), (a$_2$), (a$_3$). The only difference between the current statement and Lemma 5.1 from [12] is that, in the current conditions (a$_1$)-(a$_3$) and in (5), (6), the trajectories $t \mapsto z(t,t_0,y^0,\omega(\cdot))$ and $t \mapsto y(t,t_0,y^0,v(\eta,\cdot))$ start from the initial instant t_0 whereas, in [12] (formulae (25) and Lemma 5.1) similar trajectories start from the terminal instant T. Thus, to obtain the existence of $\{v(\eta,\cdot)\}_{\eta\in\mathbf{R}^k}$ and $M(\cdot) \in C(\mathbf{R}^k;]0,+\infty[)$ which satisfy (a$_1$),(a$_2$),(a$_3$), it suffices to repeat Section 6 from [12] modulo to the substitution $t \mapsto (T-t)$.

Combining (5), (6) and (a$_2$) (and taking into account (a$_1$) along with the form of systems (3), (4)), we obtain

$$|y(t,t_0,y^0,v(\eta,\cdot)) - y(\eta,t))| < \delta \qquad \text{for all} \ \ t \in I, \ \ \eta \in \mathbf{R}^k \tag{7}$$

Finally, using partitions of unity and arguing as in [12] (the most detailed presentation is given in [16]) we obtain the existence of a family of controls $\{\hat{v}_{\eta,\beta}(\cdot)\}_{(\eta,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}}$ which satisfies Conditions 4) and 5) of Theorem 2 and such that, for every $(\eta,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}$, the trajectory $t\mapsto y(t,t_0,y^0,\hat{v}_{(\eta,\beta)}(\cdot))$ is well-defined for all $t\in I$ and

$$|y(t,t_0,y^0,\hat{v}_{(\eta,\beta)}(\cdot))-y(t,t_0,y^0,v(\eta,\cdot))|<\delta \quad \text{for all } \ t\in I, \ \ \eta\in\mathbf{R}^k \qquad (8)$$

Combining (7), (8) with Assumption 3) of Theorem 2, and taking into account that $\delta=\frac{\varepsilon}{4}$, we obtain that the family $\{\hat{v}_{(\eta,\beta)}(\cdot)\}_{(\eta,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}}$ satisfies Condition 6) of Theorem 2 as well.

The proof of Theorem 2 is complete. This completes the proof of Theorem 1.

REFERENCES

1. Borisov V.F., Zelikin M.I. Chattering arcs in the time-optimal robots control problem // Applied Mathematics and Mechanics.– 1988. – **52**. – P. 939–946.

2. Brunovsky P. A classification of linear controllable systems // Kybernetika. – 1970. – **6**. – P. 173–188.

3. Celikovsky S., Nijmeijer H. Equivalence of nonlinear systems to triangular form: the singular case. // Systems and Control Letters. – 1996. – **27**. – P. 135–144.

4. Cheng D., Isidori A., Respondek W., Tarn T.J. Exact linearization of nonlinear systems with outputs // Math. Syst. Theory. – 1988. – **21**. – P 63–83.

5. Clarke F.H., Ledyaev Yu.S., Sontag E.D., Subbotin A.I. Asymptotic controllability implies feedback stabilization // IEEE Trans. Autom. Contr. – 1997. – **42**. – P. 1394–1407.

6. Fliess M., Levine J., Martin Ph., Rouchon R. Flatness and defect of nonlinear systems: introductory theory and examples // Int. J. Control. – 1995. – **61**. – P. 1327–1361.

7. Jakubczyk B., Respondek W. On linearization of control systems // Bull. Acad. Sci. Polonaise Ser. Sci. Math. – 1980. – **28**. – P 517–522.

8. Kokotovic P.V., Sussmann H.J. A positive real condition for global stabilization of nonlinear systems // Syst. and Contr. Lett. – 1989. – **13**. – P 125–133.

9. Kolmogorov A.N., Fomin S.V. Elements of Theory of Functions and Functional Analysis. – Translated from the First Russian Edition by Leo F. Boron, Graylock Press, Rochester, N.Y., 1957.

10. Korobov V.I. Controllability and stability of certain nonlinear systems // Differencial'nie Uravneniya. – 1973. – **9**. – P 614–619.

11. Korobov V.I., Pavlichkov S.S., Schmidt W.H. Global robust controllability of the triangular integro-differential Volterra systems // J. Math. Anal. Appl. – 2005. – **309**. – P. 743–760.

12. Korobov V.I., Pavlichkov S.S. Global properties of the triangular systems in the singular case // J. Math. Anal. Appl. – 2008. – **342**. – P. 1426-1439.

13. Kovalev A.M., Scherbak V.F. Upravlyaemost', nabludaemost', identificiruemost' dinamicheskih sistem. – Naukova Dumka, Kiev, 1980.

14. Nam K., Arapostathis A. A model reference adaptive control scheme for pure-feedback nonlinear systems // IEEE Trans. Autom. Contr. – 1988. – **33**. – P. 803–811.

15. Pavlichkov S.S. Generalized coordinate-free triangular form: global controllability and feedback triangulation // Vestn. Kharkovskogo Universiteta, Ser. Matem. Prikl. Matem., Mech. – 2007. – **790**. – P. 89–114.

16. Pavlichkov S.S. PhD Thesis, Kharkov 2001.

17. Respondek W. Global aspects of linearization, equivalence to polynomial forms and decomposition of nonlinear control systems, in: M. Fliess and M. Hazewinkel eds. // Algebraic and Geom. Meth. in Nonlinear Control Theory. – 1986. –Reidel, Dordrecht . – P. 257–284.

18. Sklyar G.M., Sklyar K.V., Ignatovich S.Yu. On the extension of the Korobov's class of linearizable triangular systems by nonlinear control systems of the class C^1 // Systems and Control Letters. – 2005. – **54**. – P. 1097-1108.

19. Sontag E.D. Mathematical Control Theory: Deterministic Finite Dimensional Systems. – NY.: Springer Verlag, 1990.

20. Tsinias J. A theorem on global stabilization of nonlinear systems by linear feedback // Syst. Contr. Lett. – 1991. – **17**. – P. 357–362.

21. Tsinias J., Partial-state global stabilization for general triangular systems // Syst. Contr. Lett. – 1995. – **24**. – P 139–145.

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