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The Taylor expansion for a-holomorphic formal power series

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Let a be a quasinilpotent element of a Banach algebra and $f(\zeta) = \sum\limits_{n=0}^{\infty} c_n\zeta^n$ a formal power series over $\mathbb{C}$. Some analogs of the Taylor expansion $f(a+h) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)h^n$ are obtained.
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1. Introduction

Holomorphic functional calculus goes far in the operator theory and its applications (for example, [1]-[5]). The essential part of this calculus are operator and Banach algebraic analogs of the Taylor expansion (see [3], Sections 3.19 and 26.4, [11], Chapter 1 and [14], Sections 10.36-10.38). Adduce one of the variants of such expansion. Let B be a Banach algebra, Ω an open set in $\mathbb{C}$, $a \in B$, the spectrum $\sigma(a)$ of a contains in Ω, and consider a function f holomorphic in Ω. Then there exists $\delta > 0$ such that for all $h \in B$, $||h|| < \delta$, $ha = ah$ we get $\sigma(a+h) \subset \Omega$ and

$$f(a+h) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)h^n, \tag{1}$$

(see [11], Chapter 1, §4, Proposition 11). At the end of 60-s S.Grabiner developed some formal power series functional (operational) calculus for quasinilpotent operators [6], [7]. Recall that an element of a Banach algebra is called quasinilpotent if its spectrum reduces to just one point $\{0\}$. S.Grabiner discussed the convergence of operator series $f(a) = \sum_{n=0}^{\infty} c_n a^n$ with respect to the various operator topologies. In the work [8] he studied the case when an operator $f(a)$ is defined not everywhere (i.e., it can be unbounded). In [6] formal power series were used to solve some finite convolution integral equations. Note, however, that there are no the Taylor expansions in the Grabiner's functional calculus. We obtain in the present paper two analogs of the Taylor formula (1), for f a formal power series with the supplementary property of a-holomorphy (see Definition 2.1). The notion of a-holomorphic power series was studied in [10] in connection with considering the holomorphic solutions of some linear differential equations in a Banach space (see also [12]). First, we consider the case the element a is nilpotent, h meets some additional condition, and f is an arbitrary h-holomorphic power series (Theorem 3.1). As the next step we put some restrictions to the growth of coefficients of a power series $f(\zeta)$ and the decrease of powers of a (Theorem 3.2).

2. Preliminaries

Let B be unital Banach algebra, $a \in B$, and $f(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n$ a formal power series over $\mathbb{C}$. Now set

$$f(za) := \sum_{n=0}^{\infty} c_n a^n z^n, z \in \mathbb{C}.$$

Then $f(za)$ is a power series with respect to z and with coefficients from B. Denote by $R_a(f)$ the radius of convergence for $f(za)$.

Definition 2.1 *We say that $f(\zeta)$ is a-holomorphic if $R_a(f) > 0$.*

Remark 2.2 Suppose that the power series f has a positive radius of convergence $R(f)$. Then such series is a-holomorphic for each $a \in B$. If $\rho(a)$ is the spectral radius of a and $|z|\rho(a) < R(f)$, then $f(za)$ stands for the result of applying the holomorphic function f to za.

Cite now some examples of the quasinilpotent elements a and the a-holomorphic formal power series with zero radius of convergence (see also [7] p.652, and [10]).

Example 2.3 Suppose that the element a is nilpotent, $a^m = 0$, then

$$f(za) = \sum_{n=0}^{m-1} c_n a^n z^n,$$

that is, each formal power series $f(\zeta)$ is a-holomorphic and $R_a(f) = +\infty$.

Example 2.4 (integration operator) Let $H = L^2(0,1)$, $B = B(H)$ is the algebra of all bounded operators on H and a is the integration operator, i.e. $(a\xi)(x) = \int_0^x \xi(y)dy, \xi \in L^2(0,1)$. It is well-known that

$$(a^n\xi)(x) = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1}\xi(y)dy, \quad ||a^n|| \leq \frac{1}{n!}, \quad n \geq 1, \tag{2}$$

and $\sigma(a) = \{0\}$ (see, for example, [9] solution of № 641). Consider now the following formal power series: $\varphi(\zeta) = \sum\limits_{n=0}^{\infty} n!\zeta^n$. It is easy to deduce from (6) and the Cauchy-Hadamard formula that φ is a-holomorphic, $R_a(\varphi) = 1$ and

$$(\varphi(za)\xi)(x) = \xi(x) + \int\limits_0^x \frac{z\xi(y)}{(1 - z(x-y))^2} dy, |z| < 1.$$

3. Main results

Theorem 3.1 *Let B be a Banach algebra, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be a formal power series over $\mathbb{C}$, $a, h \in B$ and $ha = ah$. Suppose a is nilpotent, $a^{m+1} = 0$, f is h-holomorphic and h satisfies the following condition:*

$$\exists \gamma > 1 \forall n \in \mathbb{N} : ||h^n|| \leq \gamma^n ||h^{n+1}||. \tag{3}$$

Then f is $(a+h)$-holomorphic,

$$f(z(a+h)) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(za) h^n z^n, \tag{4}$$

and $R_{a+h}(f) \geq \gamma^{-m} R_h(f)$.

Proof: Note first that $f^{(n)}(za)$ is well defined since a is nilpotent. Now show that f is $(a+h)$-holomorphic and $R_{a+h}(f) \geq \gamma^{-m} R_h(f)$. Let $k > m$, then $(a+h)^k = \sum_{j=0}^{m} C_k^j a^j h^{k-j}$ and $||(a+h)^k|| \leq \sum_{j=0}^{m} C_k^j ||a^j|| \cdot ||h^{k-j}|| \leq M \sum_{j=0}^{m} C_k^j ||h^{k-j}||$, where $M = max\{||a^j|| : 1 \leq j \leq m\}$. According to (3) we have:

$$||h^{k-j}|| \leq \gamma^{k-j} ||h^{k-j+1}|| \leq \cdots \leq \gamma^{k-j+(k-j+1)+\cdots+(k-1)} ||h^k|| = \gamma^{kj - \frac{j^2+j}{2}} ||h^k||.$$

Since $C_k^j \leq k^j$, we could obtain now

$$\sqrt[k]{M C_k^j |c_k| \cdot ||h^{k-j}||} \leq \sqrt[k]{M k^j |c_k| \cdot \gamma^{kj - \frac{j^2+j}{2}} ||h^k||} \leq M^{\frac{1}{k}} \gamma^j \gamma^{-\frac{j^2+j}{2k}} k^{\frac{j}{k}} \sqrt[k]{|c_k| \cdot ||h^k||},$$

and

$$\overline{\lim_{k\to\infty}} \sqrt[k]{M C_k^j |c_k| \cdot ||h^{k-j}||} \leq \frac{\gamma^j}{R_h(f)},$$

i.e., the power series $\sum_{k=m+1}^{\infty} C_k^j |c_k| \cdot ||a^j|| \cdot ||h^{k-j}|| z^k$ is convergent with $|z| < \gamma^{-j} R_h(f)$, $j = 0, \ldots, m$. Hence the series $\sum_{k=0}^{\infty} c_k (a+h)^k z^k$ is convergent in the disk $|z| < \gamma^{-m} R_h(f)$. Use these observations to obtain

$$\sum_{k=0}^{\infty} \sum_{n=0}^{k} C_k^n |c_k| \cdot ||a^{k-n}|| \cdot ||h^n|| \cdot |z|^k < +\infty, \tag{5}$$

where $|z| < \gamma^{-m} R_h(f)$. Hence

$$\sum_{k=0}^{\infty} c_k (a+h)^k z^k = \sum_{k=0}^{\infty} c_k \left(\sum_{n=0}^{k} C_k^n a^{k-n} h^n \right) z^k = \sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} c_k C_k^n a^{k-n} z^k \right) h^n =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=n}^{\infty} c_k k(k-1)\dots(k-(n-1)) a^{k-n} z^{k-n} \right) h^n z^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(za) h^n z^n,$$

$|z| < \gamma^{-m} R_h(f)$. The theorem is completely proved.

□

Discuss now another analog of (1).

Theorem 3.2 *Let B be a Banach algebra, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be a formal power series over $\mathbb{C}$, $a, h \in B$ and $ha = ah$. Suppose that a, h and f satisfy the following conditions:*

$$\exists M_1, M_2, M_3, \beta > 0 \quad \forall n \in \mathbb{N} : |c_n| \le M_1^n (n!)^{\beta}, \quad ||a^n|| \le \frac{M_2^n}{(n!)^{\beta}}, ||h^n|| \le \frac{M_3^n}{(n!)^{\beta}}. \tag{6}$$

Then f is $(a+h)$-holomorphic,

$$f(z(a+h)) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(za) h^n z^n,$$

and $R_{a+h}(f) \ge \frac{1}{2} \widetilde{R} \cdot (\sigma \cdot \rho)^{\frac{1}{\rho}}$, where $\widetilde{R}$ is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{c_n}{(n!)^{\beta}} z^n$, ρ and σ are the order and the type of the entire function $\psi(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} ||a^{k-j}|| \cdot ||h^j|| \right) z^k$ respectivly.

Proof: Let $\varphi_a(z) = \sum_{k=0}^{\infty} ||a^k|| z^k$ and $\varphi_h(z) = \sum_{k=0}^{\infty} ||h^k|| z^k$. It follows from (6) that both φ_a and φ_h are entire functions of a finite order less then $\frac{1}{\beta}$ and normal type (see [13], Chapter I, §2). Therefore $\psi(z) = \varphi_a(z)\varphi_h(z)$ is of finite order less then $\frac{1}{\beta}$ and normal type too ([13], Chapter I, Theorem 12). Hence

$$\overline{\lim_{k\to\infty}} \sqrt[k]{(k!)^{\beta} \left(\sum_{j=0}^{k} ||a^{k-j}|| \cdot ||h^j|| \right)} < +\infty, \tag{7}$$

(see [13], Chapter I, §2). As in the proof of Theorem 3.1, we need to check (5) on a neighborhood of zero. Since $C_k^n \le 2^k$, we deduce from (6) and (7):

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|c_k| \left(\sum_{n=0}^{k} C_k^n ||a^{k-n}|| \cdot ||h^n|| \right)} \le \overline{\lim_{k\to\infty}} \sqrt[k]{\frac{|c_k| 2^k}{(k!)^{\beta}} (k!)^{\beta} \left(\sum_{n=0}^{k} ||a^{k-n}|| \cdot ||h^n|| \right)} \le$$

$$\leq 2M_1 \overline{\lim_{k\to\infty}} \sqrt[k]{(k!)^\beta \sum_{n=0}^{k} C_k^n ||a^{k-n}|| \cdot ||h^n||} < +\infty.$$

Moreover, the inequality $\frac{1}{R_{a+h}(f)} \leq \overline{\lim_{k\to\infty}} \sqrt[k]{|c_k| \left(\sum_{n=0}^{k} C_k^n ||a^{k-n}|| \cdot ||h^n||\right)} \leq$ $2\overline{\lim_{k\to\infty}} \sqrt[k]{\frac{|c_k|}{(k!)^\beta}} \sqrt[k]{(k!)^\beta \left(\sum_{n=0}^{k} ||a^{k-n}|| \cdot ||h^n||\right)}$ shows that $R_{a+h}(f) \geq \frac{1}{2}\widetilde{R} \cdot (\sigma \cdot \rho)^{\frac{1}{\rho}}$. The theorem is proved.

□

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