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On the coincidence of the Limit Point Range and the Sum Range along a filter of filter convergent series

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We study filters $\mathcal{F}$ for which the equality $LPR = SR_{\mathcal{F}}$ holds. We characterize filters $\mathcal{F}$ for which every $\mathcal{F}$-convergent series has a null subsequence. This property with the one which we call the unbounded gap property of $\mathcal{F}$ imply that $LPR = SR_{\mathcal{F}}$. The unbounded gap property of $\mathcal{F}$ (existence of $A \in \mathcal{F}$, $A = (a_n)$ such that the sequence of $g_n = a_{n+1} - a_n$ is unbounded) is a necessary condition for this equality. The equality does not hold for any ultrafilter.

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1. Introduction

The sum range of a series on real line (the set of sums of all its convergent rearrangements) is described by the famous Riemann's theorem:

Riemann's theorem Let $\sum_{k=1}^{\infty} x_k$ be a conditionally convergent series of real numbers. Then for every $s \in \mathbb{R} \cup \{\pm\infty\}$ there exists a permutation π such that $\sum_{k=1}^{\infty} x_{\pi(k)} = s$.

There is a number of generalizations of the Riemann's theorem to more general series (vectors, functions, etc. [4]), or more general types of convergence [1], [2], [5]. The authors of [2] considered two generalizations of the usual convergence: the statistical convergence and the convergence along even numbers. They obtained the full description of the sum range along both the statistical convergence filter and the 2n-filter. Surprisingly, the simplicity of a filter does not mean the simplicity of the problem. And the description of the sum range along 2n-filter appeared to be much more complicated than along the statistical convergence filter. Besides the two usual cases of one point or the whole real line, the sum range can be an arithmetic progression $a + \lambda\mathbb{Z}$, $a, \lambda \in \mathbb{R}$ (for the statistical convergence filter) and shifted additive subgroup of the form $a + \{c_1 z_1 + \cdots + c_l z_l |\ z_k \in E,\ c_i \in \mathbb{Z},\ \sum_{k=1}^{l} c_k \text{ is even}\}$, where E is an ε-separated set (for the 2n-filter).

In this paper we continue the study originated in [2], but instead of considering concrete filters we study classes of filters for which the description of the sum range can be given explicitly. We study a class of filters $\mathcal{F}$ with the following property: *for every $\mathcal{F}$-convergent series the sum range along $\mathcal{F}$ of it coincides with its limit point range.*

Recall that a *filter* $\mathcal{F}$ on $\mathbb{N}$ is a non-empty collection of subsets of $\mathbb{N}$ satisfying the following axioms: $\emptyset \notin \mathcal{F}$; if $A, B \in \mathcal{F}$ then $A \bigcap B \in \mathcal{F}$; and for every $A \in \mathcal{F}$ if $B \supset A$ then $B \in \mathcal{F}$.

A sequence (x_n), $n \in \mathbb{N}$ in a topological space X is said to be $\mathcal{F}$-*convergent* to x if for every neighborhood U of x the set $\{n \in \mathbb{N} : x_n \in U\}$ belongs to $\mathcal{F}$. The filter convergence of series is naturally defined as filter convergence of the sequence of its partial sums, i.e. a series $\sum x_k$ is $\mathcal{F}$-*convergent* to s if the sequence $(s_n) = (\sum_{k=1}^{n} x_k)$ is $\mathcal{F}$-convergent to s (and we write $s = \mathcal{F}\text{-}\sum_k x_k$ or simply $s = \sum_{\mathcal{F}} x_k$, when there is only one possible summing index).

In particular if one takes as $\mathcal{F}$ the filter of sets with finite complements (the *Fréchet filter*), then $\mathcal{F}$-convergence coincides with the ordinary one.

The natural ordering on the set of filters on $\mathbb{N}$ is defined as follows: $\mathcal{F}_1 \succ \mathcal{F}_2$ if $\mathcal{F}_1 \supset \mathcal{F}_2$. A maximal in the natural ordering filter is called an *ultrafilter*. The Zorn lemma implies that every filter is dominated by an ultrafilter. A filter $\mathcal{F}$ on $\mathbb{N}$ is an ultrafilter if and only if for every $A \subset \mathbb{N}$ either A or $\mathbb{N} \setminus A$ belongs to $\mathcal{F}$.

A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be *free* if it dominates the Fréchet filter. Below when we say "filter" we mean a free filter on $\mathbb{N}$. In particular every ordinary convergent sequences and series will be automatically $\mathcal{F}$-convergent. More about filters, ultrafilters and their applications one can find in every modern General Topology textbook, for example in [6].

Definition 1 *A point s belongs to the sum range along $\mathcal{F}$ of the series $\sum x_k$ if there exists a permutation π such that $\sum_{\mathcal{F}} x_{\pi(k)} = s$. The set of all such points is called the* sum range along $\mathcal{F}$ of the series $\sum x_k$ *and is denoted by $SR_{\mathcal{F}}(\sum x_k)$.*

We need also the following definition from [4].

Definition 2 *A point s belongs to the limit point range of the series $\sum x_k$ if there exists a permutation π and an increasing sequence of naturals (m_k) such that $\lim_{k\to\infty} \sum_{j=1}^{m_k} x_{\pi(j)} = s$. The set of all such points is called the* limit-point range of the series $\sum x_k$ *and is denoted by $LPR(\sum x_k)$.*

It is easy to see that $LPR(\sum x_k)$ is a closed set and $SR_{\mathcal{F}}(\sum x_k) \subset LPR(\sum x_k)$. In [3] it was proved that $LPR(\sum x_k)$ is a shifted closed additive subgroup of the space in which the series lives. In particular, on the real line $LPR(\sum x_k)$ can be either $\mathbb{R}$ or $\emptyset$ or an arithmetic progression of the form $a + \lambda\mathbb{Z}$. More about series rearrangements whose terms are elements of a Banach space or of other topological linear spaces one can find in [4].

In this paper we consider series whose terms lie in the field of reals. When we write a subset of naturals $A \subset \mathbb{N}$ in the form of sequence (a_n), we mean that it is increasing.

2. Sufficient and necessary conditions for the equality $LPR = SR_{\mathcal{F}}$

It is the specificity of the usual convergence that every conditionally convergent series after rearrangement converges to every given in advance real number (the sum range is $\mathbb{R}$).

Proposition 1 *Let $\mathcal{F}$ be a filter on $\mathbb{N}$. The following conditions are equivalent*

1. *For every sequence (x_k) such that series $\sum x_k$ $\mathcal{F}$-converges $SR_{\mathcal{F}}(\sum x_k)$ is either $\mathbb{R}$ or one point $x \in \mathbb{R}$.*
2. *For every sequence (x_k) such that series $\sum x_k$ $\mathcal{F}$-converges it follows that $x_k \to_{k\to\infty} 0$.*
3. *$\mathcal{F}$ is the Fréchet filter.*

Proof. Implication (3)⇒(1) is the usual Riemann rearrangement theorem; (3)⇒(2) is the necessary condition for the usual convergence of series. To show the inverse implications (1)⇒(3) and (2)⇒(3) let us suppose the contrary. $\mathcal{F}$ is not the Fréchet filter means that there is some infinite $C \subset \mathbb{N}$ such that $A = \mathbb{N} \setminus C \in \mathcal{F}$. Denote by E the set $A \cap (C+1)$ and let (x_k) be the following sequence: x_k is 1 for $k \in E$, -1 for $k \in (E-1)$ and 0 for the rest $k \in \mathbb{N} \setminus (E \cup (E-1))$. Then $\sum x_k$ $\mathcal{F}$-converges to 0 but $SR_{\mathcal{F}}(\sum x_k) \subset \mathbb{Z}$ and $x_k \not\to_{k\to\infty} 0$. □

Thus, the sequence convergence to 0 is the characteristic property of the usual convergence of series. Let us describe a class of filters with a weaker property.

Definition 3 *A filter $\mathcal{F}$ is said to have* 1-shift property *if for every $A \in \mathcal{F}$ there is $s \in A$ such that $s+1 \in A$.*

Proposition 2 *Let $\mathcal{F}$ be a filter on $\mathbb{N}$. The following conditions are equivalent*

1. *For every sequence (x_k) such that $\sum x_k$ $\mathcal{F}$-converges there is a null subsequence (x_{k_n}).*
2. *$\mathcal{F}$ has 1-shift property.*

Proof. (2)⇒(1). Let $\sum_{\mathcal{F}} x_k = x$ and suppose that (x_k) has no null subsequence. Then there is $\delta > 0$ and $N \in \mathbb{N}$ such that $|x_k| > \delta$ for $k > N$. For $\varepsilon = \delta/4$ let us find $A = (a_k) \in \mathcal{F}$, with $a_1 > N$, such that $|x - \sum_{n=1}^{a_k} x_n| < \varepsilon$ for every $k \in \mathbb{N}$. Take $s \in A$ such that $s+1 \in A$ then we come to a contradiction:

$$\varepsilon > |x - \sum_{n=1}^{s+1} x_n| \geq |x_{s+1}| - |x - \sum_{n=1}^{s} x_n| > \delta - \varepsilon > 2\varepsilon.$$

(1)⇒(2). Suppose that there is $A = (a_k) \in \mathcal{F}$ such that $s \in A$ implies that $s+1 \notin A$, that is $a_k + 1 < a_{k+1}$ for all $k \in \mathbb{N}$. Denote $d_k = a_k - a_{k-1}$, where

$a_0 = 0$, and let (x_k) be such sequence: x_n is 1 for $n \in A$, in the case of odd d_k set $x_{a_{k-1}+1} = -1$, in the case of even d_k set $x_{a_{k-1}+1} = -2$, and set $x_n = (-1)^{n-a_{k-1}}$ for $a_{k-1}+1 < n < a_k$. Then $\sum x_k$ $\mathcal{F}$-converges but (x_k) has no null subsequence. □

Definition 4 *A filter $\mathcal{F}$ is said to have* the unbounded gap property *if there is $A \in \mathcal{F}$, $A = (a_n)$ such that the sequence of $g_n = a_{n+1} - a_n$ is unbounded.*

Theorem 1 *Let $\mathcal{F}$ be a filter which has the 1-shift property and the unbounded gap property. Then for every $\mathcal{F}$-convergent series $\sum x_k$ the equality $SR_{\mathcal{F}}(\sum x_k) = LPR(\sum x_k)$ holds.*

Proof. We proceed the same way as it was made in [2, Theorem 2.2.1.] for statistical convergence.

From the existence of a subsequence (x_{k_n}) such that $x_{k_n} \to 0$ it follows that we can select a subsequence $(x_{k_{n_i}})$ such that $\sum_{i=1}^{\infty} x_{k_{n_i}} < \infty$. Note, that absolutely convergent series does not change its sum (and hence $\mathcal{F}$-sum) under any permutation σ, so we can write

$$\sum_{\mathcal{F}} x_{\sigma(k)} = \sum_{\mathcal{F}} v_{\sigma(k)} + \sum_{k=1}^{\infty} z_k,$$

where (v_k) is the sequence (x_k) with 0 substituted for all $x_{k_{n_i}}$ and $z_k = x_k - v_k$. So without loss of generality we may assume that there are infinitely many zeros among the original series terms x_k.

Let y be an arbitrary element of $LPR(\sum x_k)$ with π and (m_k) being the permutation and the sequence corresponding to y from the definition of LPR. To obtain the permutation τ for which $y = \sum_{\mathcal{F}} x_{\tau(k)}$ let us take $A \in \mathcal{F}$, $A = (a_n)$ such that $(g_n) = (a_{n+1} - a_n)$ is unbounded and find a subsequence (g_{n_k}) such that $g_{n_k} \geq m_k - m_{k-1}$ for all $k \in \mathbb{N}$ with m_0 assigned to be 0. We arrange elements of our series in the following way:

$$\overbrace{0 + \ldots + 0}^{a_{n_1}} + x_{\pi(1)} + \ldots + x_{\pi(m_1)} + \overbrace{0 + \ldots + 0}^{a_{n_2} - a_{n_1} - m_1} +$$

$$x_{\pi(m_1+1)} + \ldots + x_{\pi(m_2)} + \underbrace{0 + \ldots + 0}_{a_{n_3} - a_{n_2} - m_2} + x_{\pi(m_2+1)} + \ldots + x_{\pi(m_3)} + \ldots.$$

For this permutation τ the series $\sum x_{\tau(k)}$ evidently $\mathcal{F}$-converge to y.
□

Theorem 2 *Let $\mathcal{F}$ be a filter which is not Fréchet. The unbounded gap property is a necessary condition in order to fulfill the equality $SR_{\mathcal{F}}(\sum x_k) = LPR(\sum x_k)$ for every $\mathcal{F}$-convergent series $\sum x_k$.*

Proof. Assuming that the unbounded gap property does not hold we construct $\mathcal{F}$-summable to 0 sequence (x_k) such that $SR_{\mathcal{F}}(\sum x_k) \neq LPR(\sum x_k)$.

The idea of the construction is the following. Let 1 be the desired limiting point which is not the $\mathcal{F}$-limit. For the series terms, to make $1 \in LPR(\sum x_k)$, we take some integers y_k from which we can make presentations of 1 as a sum of a fixed number of terms. The monotone growth of the number of these terms and the uniqueness of each presentation will result in that $1 \notin SR_{\mathcal{F}}(\sum x_k)$. Adjoining $(-y_k)$ to (y_k) and infinitely many zeros, if needed (to provide $\mathcal{F}$-convergence to 0), we will get the sequence (x_k) we need.

First, before we chose y_k, let us ascertain that $\sum_{\mathcal{F}} x_k = 0$. Since the filter we consider is not the Fréchet filter, there is $A \in \mathcal{F}$ with infinite compliment C. This allows us to rearrange our series grouping y_k and $-y_k$ in pairs. Denote $E = A \cap (C+1)$, $E = (e_1, e_2, \ldots)$ and let $x_{e_k} = y_k$, $x_{e_k - 1} = -y_k$ and the rest set zero: $x_k = 0$ for $k \in \mathbb{N} \setminus (E \cup (E-1))$. We obtain the $\mathcal{F}$-convergent to 0 series $\sum x_k$.

We take $y_k \in \mathbb{Z} \setminus \{0\}$ such that $(|y_k|)$ is a rapidly increasing sequence. The elements y_k form a tree. Its structure is as follows:

(a) for every $s \in \mathbb{N}$ there is $d(s)$ such that $y_s = \sum_{k=d(s)}^{d(s)+s} y_k$;

(b) $d(1) = 2$, $d(s+1) = d(s) + s + 1$.

(c) $y_1 = 1$;

(d) $y_{d(s)+i} = 2\sum_{k=1}^{d(s)+i-1} |y_k| + 1, \quad 0 \leq i < s;$ and from (a)

(e) $y_{d(s)+s} = -\sum_{k=d(s)}^{d(s)+s-1} y_k + y_s$.

We use the following terminology: $\{y_k\}_{k=1}^{\infty}$ is *the decomposition tree of* 1, the set $\{y_k\}_{k=d(s)}^{d(s)+s}$ is *the decomposition of* y_s, *first level of the tree* is y_1, *n-th level of the tree* is the union of decompositions of all the elements of $(n-1)$-th level. The sum of all elements of every level equals 1 for the decomposition tree of 1 and equals -1 for the decomposition tree of -1 (the set $\{-y_k\}_{k=1}^{\infty}$).

Observe that $1 \in LPR(\sum x_k)$. One can reorder and sum (x_k) for example in the following way: first take 1, then add 0, then add -1 with the decomposition of 1, then add 0, then add decomposition of -1 with the second level of the decomposition tree of 1, etc (if we do not have infinitely many zeros then we just do not add them).

Let us show that any permutation π can not make $\sum_{\mathcal{F}} x_{\pi(k)} = 1$. Supposing the contrary we find $B = (b_i) \in \mathcal{F}$ such that $\sum_{k=1}^{b_1} x_{\pi(k)} = 1$ and $\sum_{k=b_i+1}^{b_{i+1}} x_{\pi(k)} = 0$ for every $i \in \mathbb{N}$. Consider two possible cases.

The first case: there is $N \in \mathbb{N}$ such that with every element $x \in \{x_{\pi(k)}\}_{k=b_i+1}^{b_{i+1}}$ the $-x \in \{x_{\pi(k)}\}_{k=b_i+1}^{b_{i+1}}$ for all $i \geq N$. In this case the set $\{x_{\pi(k)}\}_{k=1}^{b_N}$ has to contain with every y the $-y$ in its turn. And we get $\sum_{\mathcal{F}} x_{\pi(k)} = 0$.

The second case: there are infinitely many $i \in \mathbb{N}$ for which we can find $x \in \{x_{\pi(k)}\}_{k=b_i+1}^{b_{i+1}}$ that $-x \notin \{x_{\pi(k)}\}_{k=b_i+1}^{b_{i+1}}$. Take the modulus largest $x \in \{x_{\pi(k)}\}_{k=b_i+1}^{b_{i+1}}$ that $-x \notin \{x_{\pi(k)}\}_{k=b_i+1}^{b_{i+1}}$. It is either an element of the decomposition tree of 1 or of -1. Without loss of generality let us reckon that $x = y_p$, i.e. belongs to the decomposition tree of 1. Consider the decomposition $\{y_k\}_{k=d(s)}^{d(s)+s}$ to which y_p belongs. If not all $s+1$ elements of this decomposition is contained in $\{x_{\pi(k)}\}_{k=b_i+1}^{b_{i+1}}$ then $\sum_{k=b_i+1}^{b_{i+1}} x_{\pi(k)} \neq 0$, because y_k are chosen in such a way that any sum $\sum_{k=d(s)}^{d(s)+s} \theta_k y_k$, $\theta_k \in \{0, 1, -1\}$ is ether $\pm y_s$ or is modulus-greater then $\sum_{k=1}^{d(s)-1} |y_k|$. It follows that there are infinitely many $i \in \mathbb{N}$ for which exists a decomposition that is contained in $\{x_{\pi(k)}\}_{k=b_i+1}^{b_{i+1}}$. This contradicts our assumption that $\mathcal{F}$ does not have the unbounded gap property. $\square$

Though we do not know whether the 1-shift property is a necessary condition, it is essential. To see this let us consider an example.

Proposition 3 *If $\mathcal{F}$ contains an arithmetic progression, i.e. an element A of the form $q\mathbb{N} + n$, where $q > 1$ and $0 \leq n < q$, then there is a sequence (x_k) such that series $\sum x_k$ $\mathcal{F}$-converges and $SR_{\mathcal{F}}(\sum x_k) \neq LPR(\sum x_k)$.*

Proof. Let us take $\mathcal{F}$-convergent to 0 series of the following sequence

$$(x_k) = \overbrace{0, 0, \ldots, 0}^{n}, \overbrace{1, 1, \ldots, 1, -(q-1)}^{q}, \overbrace{1, 1, \ldots, 1, -(q-1)}^{q}, \ldots .$$

For every permutation σ such that $\sum x_{\sigma(j)}$ $\mathcal{F}$-converges we can find $B \in \mathcal{F}$, $B = (b_i) \subset A$ such that $\max_{1 \leq j \leq n} \sigma(j) < b_1$, i.e. all 0-s are already summed. Then every sum $\sum_{j=1}^{b_i} x_{\sigma(j)}$ can be represented as a sum of summands with q non-zero terms in each. Every summand is the sum of m "$-(q-1)$"-s and $q - m$ "1"-s, $0 \leq m \leq q$ which is $(1-m)q$. Thus we have $SR_{\mathcal{F}}(\sum x_k) = q\mathbb{Z}$ and $1 \notin SR_{\mathcal{F}}(\sum x_k)$ but $1 \in LPR(\sum x_k)$. $\square$

In particular we have that neither the set of all odd numbers nor the set of all even numbers can belong to a filter with the property $SR_{\mathcal{F}}(\sum x_k) = LPR(\sum x_k)$. Therefore every ultrafilter does not have the studied property.

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REFERENCES

1. Bagemihl, F. ; Erdös, P. Rearrangements of C_1-summable series.// Acta Math. **92**, – 1954. – P. 35 – 53

2. Dybskiy Y.; Slutsky K., Riemann Rearrangement Theorem for some types of convergence /arXiv:math.FA/0612840v1 28 Dec 2006

3. Hadwiger, H. Über das Umordnungsproblem im Hilbertschen Raum //Math. Zeitschrift **46** – 1940. – P. 70 – 79.

4. Kadets, M. ; Kadets, V., Series in Banach spaces : conditional and unconditional convergence. Bazel: Birkhauser, – 1997 (Operator theory advances and applications; Vol.94).

5. Lorentz, G. G. ; Zeller, K. Series rearrangements and analytic sets. // Acta Math. **100** – 1958. – P. 149 – 169.

6. Todorcevic S. Topics in topology, Lecture Notes in Mathematics **1652**, Springer Verlag, Berlin, 1997.