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Banach spaces of functions with the uniform Dini property

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For a metric space K we define by $D(K)$ the space of all functions on K, whose modulus of continuity is satisfied the Dini condition at 0. We prove that $D(K)$ is dual if K is compact and $D(K)$ is separable if K is a convex compact subset of a Banach space. Other properties are studied for the special case when $K = [0, 1]$.

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1. Introduction

The famous Dini theorem says that for a continuous function $f : (0, 2\pi) \to \mathbb{R}$ satisfying the Dini condition at a point $t_0 \in (0, 2\pi)$ its Fourier series converges to $f(t_0)$ at the point t_0. Consequently if the modulus of continuity of a periodic function fulfills the Dini condition at 0, then the corresponding Fourier series converges to the original function at all points. In this paper we introduce and study the natural Banach space $D(K)$ of functions which arise from the above remark. The space definition makes sense on a general setting of continuous functions on arbitrary metric compact K. At first we describe the behaviour of Fourier series for $f \in D(\mathbb{T})$: we show that it converges uniformly, but not necessarily in the norm of $D(\mathbb{T})$. On the other hand, the theorem on Cesaro convergence holds true in $D(\mathbb{T})$ norm in its full strength. We show that $D(\mathbb{T})$ is non-reflexive and moreover contains a copy of ℓ_1. To do this we perform a study of $D(K)$ for a very special $K = \{t_1, t_2, ...\} \subset [0, 1]$, where $(t_n)_{n=1}^{\infty}$ behaves like a geometric progression. In this special case we prove that $D(K)$ has the Schur property, but has no cotype. Then we pass to the properties of $D(K)$ as a Banach space. We show that $D(K)$ is dual if K is compact and $D(K)$ is separable if K is a convex compact subset of a Banach space.

2. Basic definitions and notation

Now we introduce the following notion of the Dini space.

Definition 1 *Let K be a metric space, $f : K \longrightarrow \mathbb{R}$ be a real function. The modulus of continuity $\bar{f}(r)$ of the function f is defined by*

$$\bar{f}(r) = \sup\{|f(t) - f(\tau)| : t, \tau \in K, \rho(t,\tau) \leq r\}.$$

Denote $\|f\|_0 = \int_0^1 \frac{\bar{f}(r)}{r} dr$, $\|f\|_\infty = \sup\{|f(t)| : t \in K\}$, $\|f\| = \max\{\|f\|_0, \|f\|_\infty\}$. We call the space

$$D(K) = \{f : K \longrightarrow \mathbb{R} : \|f\| < \infty\}$$

with the norm $\|.\|$ the Dini space.

Recall the following well-known definitions.

Definition 2 *A sequence $(x_n)_{n=1}^\infty$ in a Banach space X is called a basic sequence if it is a basis for $[x_n]_{n=1}^\infty$.*

Definition 3 *Let $(e_n)_{n=1}^\infty$ be a basis for a Banach space X. Suppose that $(p_n)_{n=1}^\infty \in \mathbb{N}$ is a strictly increasing sequence of integers with $p_0 = 0$ and that $(a_n)_{n=1}^\infty$ are scalars. Then a sequence of nonzero vectors $(u_n)_{n=1}^\infty$ in X of the form*

$$u_n = \sum_{p_{n-1}+1}^{p_n} a_j e_j$$

is called a block basic sequence of $(e_n)_{n=1}^\infty$.

It is well-known that every block basic sequence $(u_n)_{n=1}^\infty$ is a basic sequence.

Definition 4 *A Banach space X has M-cotype r with constant $C > 0$ if the inequality*

$$\max\Big\{\|\sum \alpha_i x_i\| : \alpha_i = \pm 1\Big\} \geq C\Big(\sum \|x_i\|^r\Big)^{\frac{1}{r}}$$

holds for any finite collection of elements $(x_i)_{i=1}^n$.

It is well-known that M-cotype is an isomorphic invariant and the space ℓ_1 has M-cotype $r = 2$(see [2]).

3. Fourier series in $D(\mathbb{T})$

Denote by $D(\mathbb{T})$ the subspace of $D[0, 2\pi]$ consisting of functions satisfying the condition $f(0) = f(2\pi)$.

Let f be an absolutely integrable function on $[0, 2\pi]$. Then the n-th partial sum of its Fourier series at a point x equals

$$(S_n f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t + x) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2} t} dt.$$

Let $\varphi(t,x) = f(t+x) - f(x)$, then

$$(S_n f)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t,x)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt.$$

Theorem 1 (Dini's criterion) *(see [3], Section 699) Let f be a continuous function on $[0,2\pi]$ and $f(0) = f(2\pi)$. If there exists $h > 0$ such that for all $x \in [0,2\pi]$ the intergral $\int_0^h \frac{|\varphi(t,x)|}{t}dt$ converges uniformly relatively to x then the Fourier series of f coverges to f uniformly.*

If $f \in D(\mathbb{T})$ then $\varphi(t,x) \le \bar{f}(t)$. But $\int_0^1 \frac{\bar{f}(t)}{t}dt < \infty$, therefore the Fourier series of f converges to it uniformly, i.e. $\|S_n f - f\|_\infty \to 0$.

Theorem 2 *There exists a function $f \in D(\mathbb{T})$ such that the Fourier series of f does not converge to f in the norm of $D(\mathbb{T})$.*

Proof. Denote by S_n the operator of the n-th partial sum of the Fourier series. By the Banach-Steinhaus theorem, it is sufficient to show that $\sup\{\|S_n\|_0 : n \in \mathbb{N}\} = +\infty$. Fix any $n \in \mathbb{N}$ and set $\delta = \frac{\pi}{n+\frac{1}{2}}$. Then $\sin(n+\frac{1}{2})t = 0$ for all $t = k\delta$, $k \in [-n,n] \cap \mathbb{Z}$.

We will discuss the case when n is an even integer. We build on $[-\pi,\pi]$ an odd piece-wise linear function f_n as follows:

$$f_n(t) = \begin{cases} (t-2j\delta), & \text{if } t \in [2j\delta,(2j+1)\delta] \\ -[t-(2j+2)], & \text{if } t \in [(2j+1)\delta,(2j+2)\delta] \end{cases}$$

where j is from zero to $\frac{n-2}{2}$ and $f_n(t) = 0$ if $t \in [n\delta,\pi]$. Put $f_n(-t) = -f_n(t)$ for $t \ge 0$. By the construction, we have

$$\|f_n\|_0 = \int_0^{2\delta} dr + \int_{2\delta}^{1} \frac{2\delta}{r}dr = 2\delta(1+\ln\frac{1}{2\delta}).$$

Let $r \in [0,\frac{\delta}{2}]$. Notice that $\Big(f_n(t+\frac{r}{2}) - f_n(t-\frac{r}{2})\Big)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t} \ge 0$ on $[j\delta,(j+1)\delta]$ and $[f_n(t+\frac{r}{2}) - f_n(t-\frac{r}{2})] = r$ on $[j\delta+\frac{r}{2},(j+1)\delta-\frac{r}{2}]$, therefore

$$\begin{aligned}
\overline{S_n f_n}(r) &\ge (S_n f_n)(\frac{r}{2}) - (S_n f_n)(\frac{-r}{2}) \\
&\ge \frac{1}{2\pi}\sum_{j=-n}^{n-1}\int_{j\delta}^{(j+1)\delta}\Big(f_n(t+\frac{r}{2}) - f_n(t-\frac{r}{2})\Big)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt - \\
&- \frac{1}{2\pi}\int_{-(n+\frac{1}{2})\delta}^{-n\delta} r\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt - \frac{1}{2\pi}\int_{n\delta}^{(n+\frac{1}{2})\delta} r\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt \\
&\ge \frac{1}{2\pi}\sum_{j=-n}^{n-1}\int_{j\delta+\frac{r}{2}}^{(j+1)\delta-\frac{r}{2}}\Big(f_n(t+\frac{r}{2}) - f_n(t-\frac{r}{2})\Big)\frac{\sin(n+\frac{1}{2})t}{\frac{1}{2}t}dt - \frac{1}{\pi}\delta r \\
&\ge \frac{1}{2\pi}\sum_{j=-n}^{n-1} r(\delta-r)\frac{1}{|j+\frac{1}{2}|\delta} - \frac{1}{\pi}\delta r \ge M\frac{r}{\pi}\ln n,
\end{aligned}$$

where $M > 0$ is some constant not depending on n. Hence

$$\|S_n f_n\|_0 \geq M \int_0^{\frac{\delta}{2}} \frac{1}{\pi} \ln n dr + M \int_{\frac{\delta}{2}}^1 \frac{\frac{\delta}{2\pi} \ln n}{r} dr = \frac{M\delta \ln n}{2\pi}(1 + \ln \frac{2}{\delta}).$$

Therefore $\|S_n\| \geq \frac{\|S_n f_n\|_0}{\|f_n\|} \sim \ln n \to \infty$. The theorem is proved.

Cesaro's n-th average is defined as :

$$(\sigma_n f)(x) = \frac{1}{n} \sum_{j=0}^{n-1} (S_n f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t + x) F_n(t) dt$$

where $F_n(t) = \frac{1}{\pi n} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^2$ is the Fejer kernel. Notice that $\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(t) dt = 1$.

Theorem 3 (Fejer) *(see [3], Section 743) Let f be a continuous, periodic function on $\mathbb{R}$ with the period 2π. Then $(\sigma_n f)_{n=1}^{\infty}$ converges uniformly to f.*

Theorem 4 *Let $f \in D[0, 2\pi]$ and $f(0) = f(2\pi)$. Then $(\sigma_n f)_{n=1}^{\infty}$ converges to f in the norm of $D[0, 2\pi]$.*

Proof. By the previous theorem it is sufficient to show that $(\sigma_n f)_{n=1}^{\infty}$ converges to f in the norm $\|.\|_0$. Denote $\xi_n(x) = (\sigma_n f)(x) - f(x)$, then for all $x, y : |x - y| \leq r$ we have

$$\begin{aligned} |\xi_n(x) - \xi_n(y)| &= \Big| \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(f(x+t) + f(y) - f(x) - f(y+t) \Big) F_n(t) dt \Big| \\ &\leq 2\bar{f}(r) \int_{-\pi}^{\pi} F_n(t) \frac{dt}{2\pi} \\ &= 2\bar{f}(r), \end{aligned}$$

therefore $\overline{\xi_n}(r) \leq 2\bar{f}(r)$. Since $\|\xi_n\|_\infty \to 0$, the sequence of functions $\frac{\overline{\xi_n}(r)}{r}$ converges to zero almost everywhere on $[0, 1]$. By Lebesgue's theorem $\int_0^1 \frac{\overline{\xi_n}(r)}{r} \to 0$, i.e. $\|\xi_n\|_0 \to 0$.

4. The space $D(K)$ and isomorphic properties of $D[0, 1]$.

At first, we show that the space $D(K)$ is dual if K is compact, i.e. there exists a Banach space Y such that $Y^* = D(K)$. To do this, we need the following well-known theorem.

Theorem 5 *Let X be a Banach space and τ be a locally convex topology on X, which is weaker than the norm topology. If the closed unit ball B_X of X is τ-compact then X is a dual space.*

This theorem can be deduced from the bipolar theorem. In fact, if we denote $Y = \{f \in X^* : f$ is continuous in the topology $\tau\}$ considered as a subspace of X^*, then X can be identified with the space Y^* and $\sigma(X, Y)$ coincides with τ on B_X.

Theorem 6 *If K is compact then $D(K)$ is a dual space.*

Proof. On $D(K)$ consider the topology τ generated by the norm $\|.\|_\infty$. We will show that the closed unit ball $B_{D(K)}$ of $D(K)$ is τ-compact. Let $(f_n)_{n=1}^\infty \subset B_{D(K)}$ be such a sequence that converges to some function $f \in C(K)$ in the norm $\|.\|_\infty$. Then $(\bar{f}_n)_{n=1}^\infty$ converges point-wise to $\bar{f}$. By Fatou's lemma, $f \in B_{D(K)}$ and therefore $B_{D(K)}$ is τ-closed.

Suppose that $B_{D(K)}$ is not pre-compact in the topology τ. Then by Arzela's theorem

$$\exists \varepsilon > 0 : \forall \delta > 0, \exists \{t, \tau\} \subset K, \rho(t, \tau) < \delta, \exists f \in B_{D(K)}, |f(t) - f(\tau)| \geq \varepsilon.$$

Hence $\bar{f}(r) \geq \varepsilon$ for all $r \geq \delta$ and $\|f\|_0 \geq \int_\delta^1 \frac{\varepsilon}{r} dr = \varepsilon \ln \frac{1}{\delta}$. Since δ is arbitrary, there exists a function $f \in B_{D(K)}$ such that $\|f\|_0 > 1$, which is impossible. It follows that $B_{D(K)}$ is τ-compact and by the previous theorem $D(K)$ is a dual space.

By the remark made before the previous theorem, $D(K)$ is the dual space to the space $M(K)$ of all regular Borel measures in K equiped with the norm from $D(K)^*$.

Theorem 7 *For a bounded sequence $(f_n)_{n=1}^\infty$ in $D(K)$ the following three conditions are equivalent:*

1. *$(f_n)_{n=1}^\infty$ converges uniformly.*
2. *$(f_n)_{n=1}^\infty$ converges point-wise.*
3. *$(f_n)_{n=1}^\infty$ converges in the topology $\sigma(D(K), M(K))$.*

Proof. The reason for this is that $B_{D(K)}$ is a uniform compact, so every weaker Hausdorff topology on $B_{D(K)}$ coincides with the uniform one.

From this place to the end of this section we will denote

$$T = \{t_n : t_n \in [0, 1], t_\infty = 0, t_1 = 1, t_n > t_{n+1}, \lim_{n\to\infty} t_n = 0\}.$$

Obviously, the space $D(T)$ can be identified with a subspace of $D[0, 1]$.

Theorem 8 *If there exist $c \geq d > 1$ such that $d \leq \frac{t_n}{t_{n+1}} \leq c$ for all $n \in \mathbb{N}$ then on the subspace $D_0(T) = \{f \in D(T) : f(0) = 0\}$ the norm $\|.\|$ is equivalent to the norm $\|.\|_1$:*

$$\|f\|_1 = \sum_{n=1}^\infty \sup\{|f(t_j)| : j \geq n\}$$

Proof. For an arbitrary integer $j < n$ we have

$$t_{n-j} - t_{n+1-j} \geq (d-1)t_{n+1-j} \geq d^j(d-1)t_{n+1}.$$

Since $d > 1$, there exists $j_0 = j_0(d)$ such that $t_{n-j} - t_{n+1-j} > t_{n+1}$ for all $j \geq j_0$. Hence $\overline{f}(t_{n+1}) \leq 2\sup\{|f(t_i)| : i \geq n+1-j_0\}$. By the definition

$$\begin{aligned}\|f\|_0 &= \sum_{n=1}^{\infty}\int_{t_{n+1}}^{t_n}\frac{\overline{f}(r)}{r}dr \leq \sum_{n=1}^{\infty}\bar{f}(t_n)\ln\frac{t_n}{t_{n+1}} \\ &\leq \Big(\sum_{n=1}^{\infty}2\sup_{i\geq n}|f(t_i)| + \sum_{n=1}^{j_0}\bar{f}(t_n)\Big)\ln c \leq 2(j_0+1)\ln c\sum_{n=1}^{\infty}\sup_{i\geq n}|f(t_i)|.\end{aligned}$$

On the other hand,

$$\|f\|_0 = \sum_{n=1}^{\infty}\int_{t_{n+1}}^{t_n}\frac{\overline{f}(r)}{r}dr \geq \sum_{n=1}^{\infty}\bar{f}(t_{n+1})\ln\frac{t_n}{t_{n+1}} \geq \ln d\sum_{n=1}^{\infty}\sup_{i\geq n+1}|f(t_i)|$$

and

$$\begin{aligned}\|f\|_0 &\geq \int_{t_1-t_2}^{1}\frac{\overline{f}(r)}{r}dr \geq \int_{t_1-t_2}^{1}\frac{|f(t_1)-f(t_2)|}{r}dr \\ &= |f(t_1)-f(t_2)|\ln\frac{t_1}{t_1-t_2} \geq \Big(|f(t_1)| - |f(t_2)|\Big)\ln\frac{c}{c+1}.\end{aligned}$$

Therefore

$$\begin{aligned}3\|f\|_0 + \frac{\ln d}{\ln\frac{c}{c-1}}\|f\|_0 &\geq \ln d\Big(3\sum_{n=1}^{\infty}\sup_{i\geq n+1}|f(t_i)| + |f(t_1)| - |f(t_2)|\Big) \\ &\geq \ln d\sum_{n=1}^{\infty}\sup_{i\geq n}|f(t_i)|.\end{aligned}$$

Hence $\|.\|_0$ and $\|.\|_1$ are equivalent. From this it is easy to see that $\|.\|$ is equivalent to $\|.\|_1$ on $D_0(T)$.

Denote $D_0 = \{x = (x_1, x_2, ..., x_n, ..) : \|x\| = \sum_{n=1}^{\infty}\sup_{j\geq n}|x_j| < \infty\}$ and by $(e_n)_{n=1}^{\infty}$ the canonical basis of D_0. Obviously, D_0 is isomorphic to the space $D_0(K)$ and it can be considered as a subspace of $D[0,1]$.

Theorem 9 *$(e_n)_{n=1}^{\infty}$ is a basis for D_0.*

Proof. Denote by S_n the operator of the n-th partial sum:

$$S_n x = S_n(x_1, x_2, ..., x_n, ..) = (x_1, x_2, ..., x_n, 0, 0, ..)$$

and let us show that $\lim_{n\to\infty} \|x - S_n x\| = 0$.

Notice that $\left(\|x - S_n x\|\right)_{n=1}^{\infty}$ is a non-increasing sequence. Therefore, if there exists $x \in D_0$ such that $\left(\|x - S_n x\|\right)_{n=1}^{\infty}$ does not converge to zero, then there exists $\varepsilon > 0$ such that $\|x - S_n x\| \geq \varepsilon$ for all $n \in \mathbb{N}$. In detail:

$$\|x - S_n x\| = \sum_{k=n+1}^{\infty} \sup_{j\geq k} |x_j| + n \sup_{j\geq n} |x_j| \geq \varepsilon.$$

Since the first summand converges to zero as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\sup_{j\geq n} |x_j| \geq \frac{\varepsilon}{2n}$ for all $n \geq n_0$. Therefore

$$\|x\| = \sum_{n=1}^{\infty} \sup_{j\geq n} |x_j| \geq \frac{\varepsilon}{2} \sum_{n=n_0}^{\infty} \frac{1}{n} = +\infty,$$

which is impossible since $x \in D_0$.

Theorem 10 *Let $(u_n)_{n=1}^{\infty}$ be a normalized block basic sequence in D_0, $u_n = \sum_{p_{n-1}+1}^{p_n} a_j e_j$. Then there exists a subsequence $(u_{n_k})_{k=1}^{\infty}$ which is equivalent to the canonical basis of ℓ_1.*

Proof. By the definition,

$$\|u_n\| = p_{n-1} \max_{p_{n-1}+1\leq j\leq p_n} |a_j| + \sum_{j=p_{n-1}+1}^{p_n} \max_{k\geq j} |a_k|.$$

Denote $\alpha_n = \max_{p_{n-1}+1\leq j\leq p_n} |a_j|$ and $\beta_n = \sum_{j=p_{n-1}+1}^{p_n} \max_{k\geq j} |a_k|$. Then $p_{n-1}\alpha_n \leq \|u_n\| \leq p_n \alpha_n$. Since $\|u_n\| = 1$, we have that

$$\frac{1}{p_n} \leq \alpha_n \leq \frac{1}{p_{n-1}}.$$

For a given $\varepsilon > 0$ we choose $(p_{n_k})_{k=1}^{\infty}$ so that $p_{n_1} = p_1$ and $p_{n_k} \leq \varepsilon p_{n_{(k+1)}-1}$. Denote $u_{n_k} = \sum_{j=(p_{n_k-1}+1)}^{p_{n_k}} a_j e_j$ and put $M = [u_{n_k}]_{k=1}^{\infty}$. We will show that M is isomorphic to ℓ_1.

Define an operator G from M to ℓ_1 by $Gx = G\sum_{k=1}^{\infty} x_k u_{n_k} = (x_1, x_2, ...)$. Since

$$\begin{aligned} \|x\| &\geq |x_1| + \sum_{k=2}^{\infty}\Big((p_{n_k-1} - p_{n_{k-1}})\alpha_{n_k} + \beta_{n_k}\Big)|x_k| \\ &= |x_1| + \sum_{k=2}^{\infty}\Big(1 - p_{n_{k-1}}\alpha_{n_k}\Big)|x_k| \geq |x_1| + \sum_{k=2}^{\infty}(1-\varepsilon)|x_k| \\ &\geq (1-\varepsilon)\sum_{k=1}^{\infty}|x_k| = (1-\varepsilon)\|Tx\|, \end{aligned}$$

operator G is correctly defined (i.e. $Gx \in \ell_1$) and continuous. On the other hand,

$$\|x\| = \|\sum_{j=1}^{\infty} x_j u_{n_j}\| \leq \sum_{j=1}^{\infty}|x_j| = \|Tx\|.$$

It is easy to see that the set GM is dense in ℓ_1. Therefore G is an isomorphism between M and ℓ_1.

We will show that in D_0 the norm convergence coincides with the weak convergence. For that, we need the following theorem.

Theorem 11 (The Bessaga-Pełczyński selection principle) *(see [1], Prop. 1.3.10) Let* $(e_n)_{n=1}^{\infty}$ *be a basis for a Banach space* X *with dual functionals* $(e_n^*)_{n=1}^{\infty}$. *Suppose* $(x_n)_{n=1}^{\infty}$ *is a sequence in* X *such that*

1. $\inf_n \|x_n\| > 0$ *and*

2. $\lim_{n\to\infty} e_k^*(x_n) = 0$ *for all* $k \in \mathbb{N}$

Then $(x_n)_{n=1}^{\infty}$ *contains a subsequence* $(x_{n_k})_{k=1}^{\infty}$ *which is equivalent to some block basic sequence of* $(e_n)_{n=1}^{\infty}$

Theorem 12 *In the space* D_0 *the norm convergence coincides with the weak convergence.*

Proof. Assume that there exists a sequence $(x_n)_{n=1}^{\infty} \subset D_0$ such that x_n weakly converges to zero but does not converge to zero in the norm. By passing to a subsequence, we can suppose that $\inf_n \|x_n\| = \varepsilon > 0$. Denote $S = \{x_n : n \in \mathbb{N}\}$. Then $0 \notin \overline{S}^{\|\cdot\|}$ and $0 \in \overline{S}^{weak}$. By the previous theorem, S contains a subsequence $(x_{n_k})_{k=1}^{\infty}$ equivalent to some block basic sequence of $(e_n)_{n=1}^{\infty}$. Therefore without loss of generality, we can suppose that $(x_{n_k})_{k=1}^{\infty}$ is a block basic sequence of $(e_n)_{n=1}^{\infty}$.

Denote $y_k = \frac{1}{\|x_{n_k}\|}x_{n_k}$. This sequence contains a subsequence $(y_{k_j})_{j=1}^{\infty}$ equivalent to the canonical basis of ℓ_1. Therefore $(y_{k_j})_{j=1}^{\infty}$ does not converge weakly to zero. On the other hand, for any $f \in D_0^*$ we have

$$|f(y_{k_j})| = |\frac{1}{\|x_{n_{k_j}}\|}f(x_{n_{k_j}})| \leq \frac{1}{\varepsilon}|f(x_{n_{k_j}})| \to 0.$$

It means that $(y_{k_j})_{j=1}^{\infty}$ weakly converges to zero, which is impossible.

The next theorem shows that although D_0 in some sense is similar to ℓ_1, it is not isomorphic to ℓ_1.

Theorem 13 *The space D_0 does not have M-cotype r for any $r > 0$.*

Proof. Suppose that D_0 has M-cotype $r > 0$ for some $C > 0$. Consider the sequence $x_n = \frac{1}{n}e_n$. We have $\left(\sum_{j=1}^{n} \|x_j\|^r\right)^{\frac{1}{r}} = n^{\frac{1}{r}}$ and

$$\max\{\|\sum_{j=1}^{n} \alpha_j x_j\| : \alpha_j = \pm 1\} = \sum_{j=1}^{n} \frac{1}{j}.$$

Therefore $C \leq n^{-\frac{1}{r}} \sum_{j=1}^{n} \frac{1}{j} \to 0$ as $n \to \infty$. Hence $C = 0$.

5. The separability problem

Theorem 14 *If K is not pre-compact then $D(K)$ has a subspace isomorphic to ℓ_∞.*

Proof. Since K is not pre-compact, there exists $\varepsilon > 0$ and a sequence $(t_n)_{n=1}^{\infty} \subset K$ such that $\rho(t_n, t_m) > \varepsilon$ for all $n \neq m$. Denote $B_n = \{t \in K : \rho(t, t_n) < \varepsilon/4\}$. For every $\alpha = (\alpha_n)_{n=1}^{\infty} \in \ell_\infty$ we build f_α as follows:

$$f_\alpha(t) = \begin{cases} \alpha_n(\varepsilon/4 - \rho(t, t_n)), & \text{if } t \in B_n \\ 0, & \text{if } t \notin \bigcup_{n=1}^{\infty} B_n \end{cases}$$

Then $f_\alpha \in D(K)$ and it is easy to see that the map $\alpha \longmapsto f_\alpha$ is an isomorphism between ℓ_∞ and the subspace $\overline{Lin}\{f_\alpha : \alpha \in \ell_\infty\}$ of the space $D(K)$.

By this theorem, in particular, we obtain that if K is not pre-compact then the space $D(K)$ is not separable. We will prove that if K is a convex compact in a Banach space then $D(K)$ is separable.

Theorem 15 *If $K_n = \left\{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_k \in [0,1], k = \overline{1,n}\right\}$ then $D(K_n)$ is separable.*

Proof. We will show that for a given n there exists a countable subset $G_n \subset D(K_n)$ such that for every $f \in D(K_n)$ there exists $(f_k)_{k=1}^{\infty} \subset G_n$ with the following properties :

1. $\bar{f_k}(r) \le \alpha_n r + \beta_n \bar{f}(r)$, where α_n, β_n are constants depending only on n.
2. $f_k \to f$ uniformly.

From these it will follow that $f_k \to f$ in $D(K_n)$ and therefore G_n is dense in $D(K_n)$. We will show that by induction. Firstly, consider the case $n = 1$.

Let $[0,1] = \bigcup_{j=0}^{N-1} [t_j, t_{j+1}]$, where $t_j = \frac{j}{N}$. For a given $f \in D[0,1]$ we build a piece-wise linear function f_N as follows :

$$f_N(t) = \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j}(t - t_j) + f(t_j) \quad \text{if } t \in [t_j, t_{j+1}].$$

Then $f_N \to f$ uniformly on $[0,1]$ and it is easy to see that $\bar{f_N}(r) \le 5\bar{f}(r)$. We define $f_{N,k}$ as a piece-wise linear function, taking rational values at points t_j and $|f_{N,k}(t_j) - f_N(t_j)| \le \frac{1}{k}$. Then $\overline{(f_{N,k} - f_N)}(r) \le \frac{2r}{k}N$. Hence $\overline{f_{N,k}}(r) \le \frac{2r}{k}N + 5\bar{f}(r)$. Therefore the sequence $(f_{N,N})_{N=1}^{\infty}$ satisfies conditions 1 and 2. We can take G_1 as the set of all piece-wise linear functions with nodes at $\frac{j}{N}$, $N \in \mathbb{N}$, $j = 1, 2, ..., N$ and taking at points $\frac{j}{N}$ rational values.

Suppose that the set G_n exists in the space $D(K_n)$. We will show the existence of G_{n+1} in $D(K_{n+1})$. For convenience, denote $x = (y,t) \in \mathbb{R}^{n+1}$ where $y \in \mathbb{R}^n$.

Let $f \in D(K_{n+1})$. Define the function f_N as follows: on every interval $[t_j, t_{j+1}]$ if $t = \alpha t_j + (1 - \alpha t_{j+1})$ then $f_N(y,t) = \alpha\ f(y,t_j) + (1-\alpha)f(y,t_{j+1})$. It is not difficult to see that $\overline{f_N}(r) \le 6\overline{f}(r)$ and $f_N \to f$ uniformly.

By the assumption, for every t_j there exists an approximation $g_{j,k}(y)$ of the function $f(y,t_j)$ such that $g_{j,k}(y) \to f(y,t_j)$ uniformly and $\overline{g_{j,k}}(r) \le \alpha_n r + \beta_n \overline{f(.,t_j)}(r) \le \alpha_n r + \beta_n \overline{f}(r)$. Build $f_{N,k}(y,t)$:

$$f_{N,k}(y, \alpha t_j + (1-\alpha)t_{j+1}) = \alpha g_{j,k}(y) + (1-\alpha) g_{j+1,k}(y).$$

For all pair of points $(y,t), (z,\tau) \in K : \|(y,t) - (z,\tau)\| \le r$ we have

$$|f_{N,k}(y,t) - f_{N,k}(z,\tau)| \le |f_{N,k}(y,t) - f_{N,k}(z,t)| + |f_{N,k}(z,t) - f_{N,k}(z,\tau)|.$$

But

$$\begin{aligned} |f_{N,k}(z,t) - f_{N,k}(z,\tau)| &\le |f_{N,k}(z,t) - f_N(z,\tau)| + |f_{N,k}(z,\tau) - f_N(z,t)| \\ &+ |f_N(z,t) - f_N(z,\tau)| \\ &\le 2N|t-\tau| \max_{1 \le j \le N} |g_{j,k}(z) - f(z,t_j)| + \overline{f_N}(r) \\ &\le 2N|t-\tau| \max_{1 \le j \le N} |g_{j,k}(z) - f(z,t_j)| + 6\overline{f}(r) \end{aligned}$$

and the first summand

$$\begin{aligned}|f_{N,k}(y,t)-f_{N,k}(z,t)| &\leq \alpha|g_{j,k}(y,t_j)-g_{j,k}(z,t_j)|+(1-\alpha)|g_{j+1,k}(y,t_{j+1})\\ &- g_{j+1,k}(z,t_{j+1})|\\ &\leq \alpha_n r+\beta_n\overline{f}(r).\end{aligned}$$

Hence $\overline{f_{N,k}}(r) < 2rN\max_{1\leq j\leq N}|g_{j,k}(z)-f(z,t_j)|+\alpha_n r+(\beta_n+6)\overline{f}(r)$.

For a given $N>0$, $\max_{1\leq j\leq N}|g_{j,k}(z)-f(z,t_j)|\to 0$ because $g_{j,k}(z)\to f(z,t_j)$ uniformly. Therefore there exists k_N such that $\max_{1\leq j\leq N}|g_{j,k}(z)-f(z,t_j)|\leq \frac{1}{N}$. As a result, $\overline{f_{N,k_N}}(r)\leq(\alpha_n+2)r+(\beta_n+6)\overline{f}(r)$. We can take G_{n+1} as the set of all functions $g(y,t)$ piece-wise linear in the second variable with nodes in $\{\frac{j}{N}:N\in\mathbb{N}, j=1,2...N\}$ and belonging to G_n in the first variable at the nodes.

Obviously, we can take $\alpha_n=2n$ and $\beta_n=6n$. From this theorem we obtain that $D[0,1]$ is a separable dual space and thus does not contain a copy of c_0.

Corollary 1 *If $K\subset\mathbb{R}^n$ is a convex compact then $D(K)$ is separable.*

Proof. Without loss of generality we can suppose that $\mathbb{R}^n$ is the normed space with the Euclidian norm. Let $x\in\mathbb{R}^n\setminus K$. Then there exists a unique point $\bar{x}\in K$ such that $\rho(x,\bar{x})=\rho(x,K)$. Consider the map $\varphi:\mathbb{R}^n\to K$:

$$\varphi(x)=\begin{cases}x, & \text{if } x\in K\\ \bar{x}, & \text{if } x\notin K\end{cases}$$

It is easy to see that $\|\varphi(x)-\varphi(y)\|\leq\|x-y\|$ for all x,y. Therefore if $f\in D(K)$ then $f\circ\varphi$ is an extension of f on $\mathbb{R}^n$ with preservation of norm. Since K is compact, there exists $\lambda>0$ such that $K\subset\lambda K_n$. Thus every function $f\in D(K)$ can be extended to a function $F\in D(\lambda K_n)$. The space $D(\lambda K_n)$ is separable and so is $D(K)$.

Theorem 16 *If X is a Banach space and $K\subset X$ is a convex compact then the space $D(K)$ is separable.*

Proof. Consider the subspace $Y=\overline{Lin}\{x:x\in K\}$. It is separable because of compactness of K. By the Banach-Mazur theorem(see [1], Theorem 1.4.3), Y isometrically embeds into $C[0,1]$. But $C[0,1]$ has a basis, therefore we can suppose that K is a compact in a Banach space X with a basis.

Denote by S_n the operator of the n-th partial sum in X with respect to a given basis. Then $S_n\to I$ point-wise, hence S_n converges to I uniformly on K.

For a given n introduce a subspace $E_n\subset D(K)$ by $E_n=\{g:g(x)=g(S_nx)\}$. Evidently, E_n is isomorphic to the corresponding $D(S_nK)$, so by corollary 1 it is separable. Let us show that $\bigcup_{n=1}^{\infty}E_n$ is dense in $D(K)$.

Let $f \in D(K)$ and denote $f_n(x) = f(S_n x)$. Then $f_n \in E_n$ and

$$|f_n(x) - f(x)| = |f(S_n x) - f(x)| \leq \bar{f}(\|x - S_n x\|) \to 0.$$

Therefore $f_n \to f$ uniformly on K and the sequence of functions $\frac{\overline{(f_n - f)}(r)}{r}$ converges to zero almost everywhere. Denote $\alpha = \sup_n \|S_n\|$. Then

$$\begin{aligned} \Big|[f_n(x) - f(x)] - [f_n(y) - f(y)]\Big| &\leq |f(x) - f(y)| + |f_n(x) - f_n(y)| \\ &\leq \overline{f}(\|x - y\|) + \overline{f}(\|S_n x - S_n y\|) \\ &\leq \overline{f}(\|x - y\|) + \overline{f}(\alpha\|x - y\|). \end{aligned}$$

Hence $f_n \to f$ in $D(K)$ and the theorem is proved.

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