

## Strong Daugavet operators and narrow operators with respect to Daugavet centers

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A linear continuous nonzero operator  $G: X \rightarrow Y$  is a Daugavet center if every rank-1 operator  $T: X \rightarrow Y$  fulfills  $\|G + T\| = \|G\| + \|T\|$ . We introduce the notions of a  $G$ -strong Daugavet operator and a  $G$ -narrow operator which are the generalizations of the concepts of strong Daugavet and narrow operators for Daugavet centers. We also consider examples of  $G$ -narrow operators.

**Т.В. Босенко, Оператори із сильною властивістю Даугавета та вузькі оператори по відношенню до Даугаветових центрів.** Лінійний неперервний ненульовий оператор  $G: X \rightarrow Y$  називається Даугаветовим центром, якщо для кожного одновимірного оператора  $T: X \rightarrow Y$  виконується рівність  $\|G + T\| = \|G\| + \|T\|$ . У статті введені поняття оператора із  $G$ -сильною властивістю Даугавета та  $G$ -вузького оператора, які є узагальненнями понять оператора із сильною властивістю Даугавета та вузького оператора на Даугаветові центри. Розглядаються приклади  $G$ -вузьких операторів.

**Т.В. Босенко, Операторы с сильным свойством Даугавета и узкие операторы по отношению к Даугаветовым центрам.** Линейный непрерывный ненулевой оператор  $G: X \rightarrow Y$  называется Даугаветовым центром, если для любого одномерного оператора  $T: X \rightarrow Y$  выполняется равенство  $\|G + T\| = \|G\| + \|T\|$ . В статье введены понятия оператора с  $G$ -сильным свойством Даугавета и  $G$ -узкого оператора, которые являются обобщениями понятий оператора с сильным свойством Даугавета и узкого оператора на Даугаветовы центры. Рассматриваются примеры  $G$ -узких операторов.

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### 1. Introduction

In the present paper we deal with real Banach spaces and denote them  $X$ ,  $Y$  or  $E$ . We use the notation  $L(X, Y)$  for the space of all linear continuous operators  $T: X \rightarrow Y$ . Throughout the paper we use the word "operator" in the sense of "linear continuous operator". We denote the identity operator on a Banach space by the symbol  $\text{Id}$ .

Banach space  $X$  is said to have *the Daugavet property* [5] if every rank-1 operator  $T: X \rightarrow X$  fulfills the equation

$$\|\text{Id} + T\| = 1 + \|T\|, \quad (1)$$

which is known as *Daugavet equation*.

The Daugavet equation theory has been rapidly developing during the past two decades (see [5], [9], [11], [12]). Examples of spaces with the Daugavet property include  $C(K)$  where  $K$  is a compact without isolated points [4],  $L_1(\mu)$  and  $L_\infty(\mu)$  where  $\mu$  has no atoms [8], and some Banach algebras (see [5], [11], [12]). The notions of a strong Daugavet operator and a narrow operator were introduced in [7] and take an important place in the Daugavet equation theory.

**Definition 1** *An operator  $T: X \rightarrow E$  is said to be a strong Daugavet operator if for every  $x_0, y_0$  from the unit sphere of  $X$  and for every  $\varepsilon > 0$  there is a  $z$  from the unit ball of  $X$  such that  $\|T(x_0 - z)\| < \varepsilon$  and  $\|z + y_0\| > 2 - \varepsilon$ .*

**Definition 2** *Let  $X$  have the Daugavet property. An operator  $T: X \rightarrow E$  is said to be narrow if for every  $x^* \in X^*$  the operator  $T \tilde{+} x^*: X \rightarrow E \oplus_1 \mathbb{R}$ ,  $(T \tilde{+} x^*)x = (Tx, x^*(x))$  is a strong Daugavet operator.*

It is easy to see that every narrow operator on  $X$  is a strong Daugavet operator. Every strong Daugavet operator  $T: X \rightarrow X$  satisfies (1). The class of narrow operators on a Banach space with the Daugavet property appeared to be large; in particular, weakly compact operators, operators not fixing a copy of  $\ell_1$ , strong Radon-Nikodým operators and hereditary *SCD*-operators on Banach spaces with the Daugavet property are narrow [1], [7].

In [2] it was suggested to generalize the Daugavet equation theory with the help of the following concept.

**Definition 3** *An operator  $G \in L(X, Y) \setminus \{0\}$  is called a Daugavet center if every rank-1 operator  $T: X \rightarrow Y$  fulfills the identity*

$$\|G + T\| = \|G\| + \|T\|. \quad (2)$$

Thus, the Daugavet equation theory extends to the spaces which have a Daugavet center acting from them, so called Daugavet domains, and to the spaces which have a Daugavet center acting into them, so called Daugavet ranges. Like spaces with the Daugavet property, Daugavet domains and Daugavet ranges are non-reflexive, contain subspaces isomorphic to  $\ell_1$ , do not have an unconditional

basis (countable or uncountable) and cannot have the Radon-Nikodým property ([2], [5], [10]). However, it was shown in [3] that a Daugavet domain, a Daugavet range and a space with the Daugavet property are three different notions indeed.

The aim of this paper is to find appropriate generalizations of the concepts of strong Daugavet and narrow operators for Daugavet centers (see Definition 4 and Definition 6 of  $G$ -strong Daugavet and  $G$ -narrow operators) in such a way that the basic facts about the original classes can be transferred to these generalizations.

In Section 2 we generalize some results of [7]. Namely, we find a collection  $\mathcal{D}_G(X)$  of subsets in  $X$  such that  $T \in L(X, E)$  is a  $G$ -strong Daugavet operator if and only if  $T$  is unbounded from below on every  $A \in \mathcal{D}_G(X)$  (see Definition 5). We also present a geometrical characterization of a  $G$ -narrow operator (see Proposition 4).

In Section 3 we generalize some results of [1] and [6], namely we show that for every Daugavet center  $G: X \rightarrow Y$  all the  $SCD$ -operators on  $X$  are  $G$ -strong Daugavet operators (see Proposition 5) and all the hereditary  $SCD$ -operators on  $X$  are  $G$ -narrow (see Corollary 2). As a consequence we obtain that weakly compact operators, operators not fixing a copy of  $\ell_1$ , strong Radon-Nikodým operators on  $X$  are  $G$ -narrow as well.

Section 4 is devoted to examples of  $G$ -narrow operators. We show that the classes of narrow and  $G$ -narrow operators coincide for some Daugavet centers  $G$ , e.g. if  $G$  is a surjective isometry acting in a space with the Daugavet property. We also prove that there exist Daugavet centers  $G$  for which the classes of narrow and  $G$ -narrow operators do not coincide. We give a simple but surprising example of a narrow operator which is a Daugavet center (see Example 1).

In this work we use the following notation. The symbol  $B_X$  stands for the closed unit ball of  $X$  and  $S_X$  denotes its unit sphere. For an element  $x^* \in X^*$ , a slice of a (nonempty) bounded closed convex set  $A \subset X$  is given by

$$S(A, x^*, \alpha) := \{x \in A: x^*(x) > \sup_{a \in A} x^*(a) - \alpha\}$$

where  $0 < \alpha < \sup x^*(A)$ . If  $A \in X^*$  and  $x \in X$  then we will say that the slice

$$S(A, x, \alpha) := \{z^* \in A: z^*(x) > \sup_{y^* \in A} y^*(x) - \alpha\}$$

is a  $w^*$ -slice of  $A$ .

In the present paper we deal with Daugavet centers  $G$  with  $\|G\| = 1$  but our results extend easily to the case  $\|G\| \neq 1$  because if operators  $G$  and  $T$  satisfy the equation (2) then  $\|aG + bT\| = a\|G\| + b\|T\|$  holds true for every  $a, b \geq 0$ .

In the sequel we are going to use the following geometrical characterization of a Daugavet center.

**Theorem 1 ([2], Theorem 2.1)** *For an operator  $G: X \rightarrow Y$  with  $\|G\| = 1$  the following assertions are equivalent:*

- (i)  $G$  is a Daugavet center.

- (ii) For every  $y_0 \in S_Y$ ,  $x_0^* \in S_{X^*}$  and  $\varepsilon > 0$  there is an  $x \in S(B_X, x_0^*, \varepsilon)$  with  $\|Gx + y_0\| > 2 - \varepsilon$ .
- (iii) For every  $x_0^* \in S_{X^*}$  and every  $w^*$ -slice  $S(B_{Y^*}, y_0, \varepsilon_0)$  there is another  $w^*$ -slice  $S(B_{Y^*}, y_1, \varepsilon_1) \subset S(B_{Y^*}, y_0, \varepsilon_0)$  such that for every  $y^* \in S(B_{Y^*}, y_1, \varepsilon_1)$  the inequality  $\|G^*y^* + x_0^*\| > 2 - \varepsilon_0$  holds true.

## 2. $G$ -strong Daugavet operators and $G$ -Narrow operators

In this section we generalize some results from [7].

**Definition 4** Let  $G \in S_{L(X,Y)}$ . An operator  $T: X \rightarrow E$  is said to be a  $G$ -strong Daugavet operator if for every  $x_0 \in S_X$ ,  $y_0 \in S_Y$  and  $\varepsilon > 0$  there is a  $z \in B_X$  such that  $\|T(x_0 - z)\| < \varepsilon$  and  $\|Gz + y_0\| > 2 - \varepsilon$ .

Note that in the above definition it is not important which space a  $G$ -strong Daugavet operator acts into, but  $\|Tx\|$  for every  $x \in X$  is significant. Therefore, following [7] we say that operators  $T_1: X \rightarrow E_1$  and  $T_2: X \rightarrow E_2$  are *equivalent* if  $\|T_1x\| = \|T_2x\|$  for every  $x \in X$ . The symbol  $\mathcal{OP}(X)$  denotes the class of all operators on  $X$  with the convention that equivalent operators are identified. We use the notation  $\mathcal{SD}(X)$  for the class of all strong Daugavet operators  $T \in \mathcal{OP}(X)$ , and  $\mathcal{NAR}(X)$  for the class of all narrow operators  $T \in \mathcal{OP}(X)$ . It was shown in [7] that  $\mathcal{NAR}(X) \neq \emptyset$  if and only if  $X$  has the Daugavet property. In the present paper we denote the class of all  $G$ -strong Daugavet operators  $T \in \mathcal{OP}(X)$  by  $\mathcal{SD}(G, X)$ .

**Lemma 1** Let  $G \in S_{L(X,Y)}$ . If  $T: X \rightarrow Y$  is a  $G$ -strong Daugavet operator then  $\|G + T\| = 1 + \|T\|$ .

*Proof.* We assume without loss of generality that  $\|T\| = 1$ . Let  $\varepsilon > 0$  and  $x \in S_X$  such that  $\|Tx\| \geq 1 - \varepsilon$ . Put  $y = Tx/\|Tx\|$ . By Definition 4 there is a  $z \in B_X$  such that  $\|T(x - z)\| < \varepsilon$  and  $\|Gz + y\| > 2 - \varepsilon$ . Hence

$$2 - \varepsilon < \|Gz + y\| \leq \|Gz + Tx\| + \varepsilon \leq \|Gz + Tz\| + 2\varepsilon.$$

Then

$$\|Gz + Tz\| \geq 2 - 3\varepsilon$$

which proves the lemma.  $\square$

Our next goal is to show that if  $G \in S_{L(X,Y)}$  is a Daugavet center then every finite-rank operator on  $X$  is a  $G$ -strong Daugavet operator. For future reference we record the following lemma.

**Lemma 2 ([7], Lemma 3.9)** For every  $\tau > 0$  and every pair of positive numbers  $a, b$  there is a  $\delta > 0$  such that if  $v, x \in B_X$  and  $\|x + v\| > 2 - \delta$ , then  $\|ax + bv\| > a + b - \tau$ .

Now we prove a characterization of a Daugavet center. We use the idea of Lemma 3 of [10] in its proof.

**Lemma 3** For  $G \in S_{L(X,Y)}$  the following assertions are equivalent:

- (i)  $G$  is a Daugavet center.
- (ii) For every  $\varepsilon > 0$ ,  $y \in S_Y$  and every nonvoid relatively weakly open subset  $U$  of  $B_X$  there exists an  $x \in U$  such that  $\|Gx + y\| > 2 - \varepsilon$ .

*Proof.*

(i)  $\Rightarrow$  (ii) It was shown in [10] that there exist slices  $S_1, S_2, \dots, S_n$  of  $B_X$  such that a convex combination  $\sum_{i=1}^n \lambda_i S_i \subset U$ . Using Lemma 2 and item (ii) of Theorem 1 we find an  $x_1 \in S_1$  with  $\|\lambda_1 Gx_1 + y\| > \lambda_1 + 1 - \varepsilon$ . Analogously there is an  $x_2 \in S_2$  such that  $\|\lambda_2 Gx_2 + \lambda_1 Gx_1 + y\| > \lambda_2 + \lambda_1 + 1 - \varepsilon$ . Continuing in the same way we find an  $x_n \in S_n$  with  $\|\lambda_n Gx_n + \lambda_{n-1} Gx_{n-1} + \dots + \lambda_1 Gx_1 + y\| > \lambda_n + \lambda_{n-1} + \dots + \lambda_1 + 1 - \varepsilon$ . Denote  $x := \sum_{i=1}^n \lambda_i x_i$ . Then  $x \in \sum_{i=1}^n \lambda_i S_i \subset U$  and  $\|Gx + y\| > 2 - \varepsilon$ .

The implication (ii)  $\Rightarrow$  (i) follows easily from Theorem 1 because every slice  $S(B_X, x^*, \varepsilon)$  of is a nonvoid relatively weakly open set.  $\square$

**Proposition 1** If  $G \in S_{L(X,Y)}$  is a Daugavet center then every finite-rank operator on  $X$  is a  $G$ -strong Daugavet operator.

*Proof.* Let  $\varepsilon > 0$ ,  $x_0 \in S_X$  and  $y_0 \in S_Y$ . Consider a finite-rank  $T \in L(X, E)$ . Then  $T$  is continuous, acting from  $X$  with the weak topology into  $E$  with the norm topology. Then  $\{z \in B_X: \|Tx_0 - Tz\| < \varepsilon\}$  is a nonvoid relatively weakly open set. Hence Lemma 3 implies that there exists  $z_0 \in B_X$  with  $\|T(x_0 - z_0)\| < \varepsilon$  and  $\|Gz_0 + y_0\| > 2 - \varepsilon$ . Then by Definition 4  $T$  is a  $G$ -strong Daugavet operator.  $\square$

Now we show a relationship between  $G$ -strong Daugavet operators and the following collection of sets.

**Definition 5** Let  $G \in S_{L(X,Y)}$ . For every  $x \in S_X$ ,  $y \in S_Y$  and every  $\varepsilon > 0$  let us define

$$D_G(x, y, \varepsilon) := \{z \in X : \|Gz + Gx + y\| > 2 - \varepsilon \ \& \ \|z + x\| < 1 + \varepsilon\}.$$

By  $\mathcal{D}_G(X)$  we denote the collection of all sets  $D_G(x, y, \varepsilon)$  where  $x \in S_X$ ,  $y \in S_Y$  and  $\varepsilon > 0$ .

Let  $\mathcal{N}$  be a collection of subsets in  $X$ . Following [7] in our paper we use the symbol  $\mathcal{N}^\sim$  for the class of all  $T \in \mathcal{OP}(X)$  unbounded from below on every  $A \in \mathcal{N}$ , i.e.,

$$\forall A \in \mathcal{N} \ \forall \varepsilon > 0 \ \exists x \in A: \|Tx\| < \varepsilon.$$

We also use the notation

$$U_{T,\varepsilon} = \{x \in X: \|Tx\| < \varepsilon\}$$

for the tube determined by  $T \in \mathcal{OP}(X)$  and  $\varepsilon > 0$ .

**Proposition 2**  $\mathcal{SD}(G, X) = \mathcal{D}_G(X)^\sim$ .

*Proof.*  $T \in \mathcal{D}_G(X)^\sim$  if and only if for every  $x \in S_X$ ,  $y \in S_Y$  and  $\varepsilon > 0$  there exists a  $z \in D_G(x, y, \varepsilon)$  such that  $\|Tz\| < \varepsilon$ . This is equivalent to the following assertion: for every  $x \in S_X$ ,  $y \in S_Y$  and  $\varepsilon > 0$  there is a  $v \in x + U_{T, \varepsilon}$  such that  $\|v\| < 1 + \varepsilon$  and  $\|Gv + y\| > 2 - \varepsilon$ . Then by Definition 4  $T$  is a  $G$ -strong Daugavet operator.  $\square$

Let us recall how the operation of addition is defined on  $\mathcal{OP}(X)$  [7]. The  $\sim$ sum of  $T_1: X \rightarrow E_1$  and  $T_2: X \rightarrow E_2$  is

$$T_1 \tilde{+} T_2: X \rightarrow E_1 \oplus_1 E_2, \quad x \mapsto (T_1x, T_2x);$$

i.e.,

$$\|(T_1 \tilde{+} T_2)x\| = \|T_1x\| + \|T_2x\|.$$

For non-empty subsets  $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{OP}(X)$  their  $\sim$ sum is

$$\mathcal{M}_1 \tilde{+} \mathcal{M}_2 = \{T_1 \tilde{+} T_2: T_1 \in \mathcal{M}_1, T_2 \in \mathcal{M}_2\}$$

and their  $\sim$ difference is

$$\mathcal{M}_2 \tilde{-} \mathcal{M}_1 = \{T \in \mathcal{OP}(X): T \tilde{+} T_1 \in \mathcal{M}_2 \text{ whenever } T_1 \in \mathcal{M}_1\}.$$

**Definition 6** Let  $G \in S_{L(X, Y)}$ . Define the class of  $G$ -narrow operators by  $\mathcal{NAR}(G, X) = \mathcal{SD}(G, X) \tilde{-} X^*$ .

Thus, an operator  $T: X \rightarrow E$  is  $G$ -narrow if for every  $x^* \in X^*$  the  $\sim$ sum  $T \tilde{+} x^*$  is a  $G$ -strong Daugavet operator. Definition 6, Lemma 1 and Proposition 1 imply the following facts:

- If  $G \in S_{L(X, Y)}$  is a Daugavet center then every finite-rank operator on  $X$  is  $G$ -narrow.
- If for an operator  $G$  there exists at least one  $G$ -narrow operator then  $G$  is a Daugavet center.

Our next goal is to prove Proposition 4 which is a geometrical characterization of a  $G$ -narrow operator. For this purpose we use the following fact.

**Proposition 3 ([7], Proposition 2.12)** Let  $\mathcal{M} \subset \mathcal{OP}(X)$  and let  $\mathcal{N}$  be a collection of subsets in  $X$ . Then  $\mathcal{N}^\sim \tilde{-} \mathcal{M} = \mathcal{N}_1^\sim$ , where  $\mathcal{N}_1$  consists of all intersections of the form  $U_{T, \varepsilon} \cap A$ ,  $T \in \mathcal{M}$ ,  $A \in \mathcal{N}$ ,  $\varepsilon > 0$ .

**Lemma 4** Let  $G \in S_{L(X, Y)}$  and  $T \in \mathcal{NAR}(G, X)$ . Then for every  $x \in S_X$ ,  $y \in S_Y$ ,  $\varepsilon > 0$  and every slice  $S = S(B_X, x^*, \alpha)$  containing  $x$  there is a  $v \in S$  such that  $\|Gv + y\| > 2 - \varepsilon$  and  $\|T(x - v)\| < \varepsilon$ .

*Proof.* Suppose  $T$  is narrow. Since  $x \in S$  then there is an  $\varepsilon_1 > 0$  such that  $x^*(x) > 1 - \alpha + \varepsilon_1$ . Proposition 3 implies that for every  $\delta > 0$  there is a  $u \in U_{x^*,\delta} \cap D_G(x, y, \delta)$  with  $\|Tu\| < \delta$ . This means that  $|x^*(u)| < \delta$ ,  $\|Tu\| < \delta$ ,  $\|x + u\| < 1 + \delta$  and  $\|Gx + Gu + y\| > 2 - \delta$ . Put  $v := (x + u)/\|x + u\|$  then

$$x^*(v) > \frac{1}{1 + \delta}(1 - \alpha + \varepsilon_1 - \delta)$$

and

$$\|T(x - v)\| = \left\| T\left(x - \frac{x + u}{\|x + u\|}\right) \right\| \leq \frac{\| \|x + u\| - 1 \|Tx\| + \|Tu\|}{\|x + u\|} < \frac{\delta(\|T\| + 1)}{1 - \delta}$$

and

$$\|Gv + y\| \geq \|Gx + Gu + y\| - \left\| G(x + u) - \frac{G(x + u)}{\|x + u\|} \right\| > 2 - 2\delta.$$

If  $\delta$  is small enough then  $v$  satisfies our requirements. □

**Lemma 5** *Let  $T \in \mathcal{NAR}(G, X)$ .*

- (a) *Let  $S_1, \dots, S_n$  be a finite collection of slices and  $U \subset B_X$  be a convex combination of these slices, i.e., there are  $\lambda_k \geq 0$ ,  $k = 1, \dots, n$ ,  $\sum_{k=1}^n \lambda_k = 1$ , such that  $\lambda_1 S_1 + \dots + \lambda_n S_n = U$ . Then for every  $\varepsilon > 0$ , every  $y \in S_Y$  and every  $w \in U$  there is a  $u \in U$  such that  $\|Gu + y\| > 2 - \varepsilon$  and  $\|T(w - u)\| < \varepsilon$ .*
- (b) *The same conclusion is true if  $U \subset B_X$  is a relatively weakly open set.*

*Proof.* (a) Pick  $x_j \in S_j$  such that  $\lambda_1 x_1 + \dots + \lambda_n x_n = w$ . We can assume that  $x_j \in S_X$  since every  $\hat{x} \in S_j$  can be represented as a convex combination of some  $\hat{x}_1, \hat{x}_2 \in S_j \cap S_X$ . Applying repeatedly Lemma 4 and Lemma 2 with sufficiently small  $\varepsilon_j$  to  $S_j$ ,  $x_j \in S_j$  and

$$y_j = \left( y + \sum_{k=1}^{j-1} \lambda_k Gv_k \right) / \left\| y + \sum_{k=1}^{j-1} \lambda_k Gv_k \right\|,$$

we select  $v_k \in S_k$  with  $\|T(x_k - v_k)\| < \varepsilon_k$ ,  $k = 1, \dots, n$ , in such a way that for every  $j = 1, \dots, n$

$$\left\| y + \sum_{k=1}^j \lambda_k Gv_k \right\| > 1 + \sum_{k=1}^j \lambda_k (1 - \varepsilon).$$

Then  $u := \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$  is as required.

(b) It was proved in [10] that for every relatively weakly open set  $U \subset B_X$  there is a convex combination  $V$  of slices of  $B_X$  such that  $V \subset U$ . In fact, it is easy to see that the union of all  $V \subset U$  which are convex combinations of slices,

is dense in  $U$ . Hence for every  $w \in U$  and every  $\varepsilon > 0$  there exists a convex combination  $V_0$  of slices of  $B_X$  and there is a  $v \in V_0$  such that

$$\|w - v\| < \frac{\varepsilon}{2\|T\|}.$$

Then (a) implies that for every  $y \in S_Y$  there is a  $u \in V_0$  such that  $\|T(u-v)\| < \varepsilon/2$  and  $\|Gu + y\| > 2 - \varepsilon$ . So we have

$$\|T(w - u)\| \leq \|T(w - v)\| + \|T(v - u)\| < \varepsilon$$

which gives the needed result.  $\square$

**Proposition 4** *Let  $G \in S_{L(X,Y)}$  be a Daugavet center. For  $T \in L(X,E)$  the following assertions are equivalent:*

- (i)  $T$  is  $G$ -narrow.
- (ii) For every  $x \in S_X$ ,  $y \in S_Y$ ,  $\varepsilon > 0$  and every slice  $S = S(B_X, x^*, \alpha)$  containing  $x$  there is a  $v \in S$  such that  $\|Gv + y\| > 2 - \varepsilon$  and  $\|T(x - v)\| < \varepsilon$ .

*Proof.* It only remains to prove (ii)  $\Rightarrow$  (i). Remark that an operator satisfying (ii) also fulfills the conclusion of Lemma 5. Let  $x_0^* \in X^*$ ,  $x \in S_X$ ,  $y \in S_Y$  and  $\varepsilon > 0$ . Consider the relatively weakly open set

$$U := \{z \in B_X: |x_0^*(x - z)| < \varepsilon/2\}$$

then  $x \in U$ . By Lemma 5 there is a  $u \in U$  with  $\|Gu + y\| > 2 - \varepsilon/2$  and  $\|T(x - u)\| < \varepsilon/2$ . By Definition 4  $T \tilde{+} x_0^*$  is a  $G$ -strong Daugavet operator. Then  $T$  is  $G$ -narrow.  $\square$

### 3. Hereditary $SCD$ -operators are $G$ -narrow

In this section we generalize some results from [1] and [6]. A bounded convex set  $A \subset X$  is a slicely countably determined ( $SCD$ ) set if there is a sequence  $\{S_n: n \in \mathbb{N}\}$  of slices of  $A$  such that  $A \subset \overline{\text{conv}} B$  whenever  $B \subseteq A$  intersects all the  $S_n$ 's. A linear continuous operator  $T: X \rightarrow E$  is called an  $SCD$ -operator if  $T(B_X)$  is an  $SCD$  set, and a hereditary  $SCD$ -operator if every bounded convex subset of  $T(B_X)$  is an  $SCD$  set. All the operators not fixing a copy of  $\ell_1$  and strong Radon-Nikodým operators are proved to be hereditary  $SCD$ -operators [1].

Our aim in this section is to show that for every Daugavet center  $G: X \rightarrow Y$  all the  $SCD$ -operators on  $X$  are  $G$ -strong Daugavet operators and all the hereditary  $SCD$ -operators on  $X$  are  $G$ -narrow. First we prove a characterization of a Daugavet center.

Consider a  $G \in S_{L(X,Y)}$ . Denote  $K(Y^*)$  the weak\*-closure of the set of all extreme points of  $B_{Y^*}$ .

**Remark 1** *Every  $w^*$ -slice  $S(B_{Y^*}, y_0, \varepsilon_0)$  satisfies*

$$S(B_{Y^*}, y_0, \varepsilon_0) \cap K(Y^*) \neq \emptyset.$$

For  $x^* \in X^*$  and  $\varepsilon > 0$  we write  $S'(x^*, \varepsilon) = \{x \in B_X: x^*(x) > 1 - \varepsilon\}$ . Remark that  $S'(x^*, \varepsilon) \neq \emptyset$  if and only if  $\|x^*\| > 1 - \varepsilon$ . For every  $\varepsilon > 0$  and every slice  $S$  of  $B_X$  we denote

$$A(G, S, \varepsilon) = \{y^* \in K(Y^*): S \cap S'(G^*y^*, \varepsilon) \neq \emptyset\}.$$

**Remark 2** For every  $\varepsilon > 0$  and every slice  $S$  of  $B_X$  the set  $A(G, S, \varepsilon)$  is relatively weak\*-open in  $K(Y^*)$ .

**Lemma 6** For a  $G \in S_{L(X,Y)}$  the following assertions are equivalent:

- (i)  $G$  is a Daugavet center.
- (ii) For every  $\varepsilon > 0$ , every  $y \in S_Y$  and every slice  $S$  of  $B_X$  there is a  $y^* \in A(G, S, \varepsilon)$  such that  $y \in S'(y^*, \varepsilon)$ .
- (iii) For every  $\varepsilon > 0$  and every slice  $S$  of  $B_X$  the set  $A(G, S, \varepsilon)$  is weak\*-dense in  $K(Y^*)$ .
- (iv) For every  $\varepsilon > 0$  and every sequence  $\{S_n: n \in \mathbb{N}\}$  of slices of  $B_X$  the set  $\bigcap_{n \in \mathbb{N}} A(G, S_n, \varepsilon)$  is weak\*-dense in  $K(Y^*)$ .

*Proof.* (i)  $\Rightarrow$  (ii) Pick an  $\varepsilon > 0$ , a  $y \in S_Y$  and an  $S$ . There exist  $x_0^* \in S_{X^*}$  and  $\delta > 0$  such that  $S = S(B_X, x_0^*, \delta)$ . Denote  $\varepsilon_0 := \min\{\varepsilon, \delta\}$ .

Consider the  $w^*$ -slice  $S(B_{Y^*}, y, \varepsilon)$ . If  $G$  is a Daugavet center then by item (iii) of Theorem 1 there is a  $w^*$ -slice  $S(B_{Y^*}, y_1, \varepsilon_1) \subset S(B_{Y^*}, y, \varepsilon)$  such that every  $y^* \in S(B_{Y^*}, y_1, \varepsilon_1)$  satisfies  $\|G^*y^* + x_0^*\| > 2 - \varepsilon_0/2$ .

According to Remark 1 we pick a  $y^* \in S(B_{Y^*}, y_1, \varepsilon_1) \cap K(Y^*)$ . Consider  $S(B_X, G^*y^* + x_0^*, \varepsilon_0/2) = \{x \in B_X: G^*y^*(x) + x_0^*(x) > \|G^*y^* + x_0^*\| - \varepsilon_0/2\}$ . Then every  $x \in S(B_X, G^*y^* + x_0^*, \varepsilon_0/2)$  fulfills  $G^*y^*(x) + x_0^*(x) > 2 - \varepsilon_0$ . But  $G^*y^*(x) \leq 1$  and  $x_0^*(x) \leq 1$ , hence we have  $G^*y^*(x) > 1 - \varepsilon_0 \geq 1 - \varepsilon$  and  $x_0^*(x) > 1 - \varepsilon_0 \geq 1 - \delta$ . This means that  $x \in S \cap S'(G^*y^*, \varepsilon)$ . Consequently  $y^* \in A(G, S, \varepsilon)$ . And, since  $y^* \in S(B_{Y^*}, y_1, \varepsilon_1) \subset S(B_{Y^*}, y, \varepsilon)$  then  $y^*(y) > 1 - \varepsilon$ , hence  $y \in S'(y^*, \varepsilon)$ .

(ii)  $\Rightarrow$  (i) Pick an  $\varepsilon > 0$ , a  $y \in S_Y$  and an  $x^* \in S_{X^*}$ . Then there is a  $y^* \in A(G, S(B_X, x^*, \varepsilon), \varepsilon/2)$  such that  $y \in S'(y^*, \varepsilon/2)$ . Hence there exist an  $x \in S(B_X, x^*, \varepsilon)$  such that  $y^*(Gx) = (G^*y^*)(x) > 1 - \varepsilon/2$  and so

$$\|Gx + y\| \geq \|Gx + y\| \|y^*\| \geq |y^*(Gx) + y^*(y)| > 1 - \varepsilon/2 + 1 - \varepsilon/2 = 2 - \varepsilon.$$

Then by item (ii) of Theorem 1  $G$  is a Daugavet center.

(ii)  $\Rightarrow$  (iii) To prove that  $A(G, S, \varepsilon)$  is weak\*-dense in  $K(Y^*)$ , it is sufficient to show that the weak\* closure of  $A(G, S, \varepsilon)$  contains every extreme point  $y^*$  of  $B_{Y^*}$ . Since  $w^*$ -slices form a base of neighborhoods of extreme points in  $B_{Y^*}$ , we need to prove that every  $w^*$ -slice  $S(B_{Y^*}, y, \delta)$  with  $\delta \in (0, \varepsilon)$  intersects  $A(G, S, \varepsilon)$ , i.e. that there is a point  $y^* \in A(G, S, \varepsilon)$  such that  $y^* \in S(B_{Y^*}, y, \delta)$ . But we know that there is a point  $y^* \in A(G, S, \delta) \subseteq A(G, S, \varepsilon)$  such that  $y \in S'(y^*, \delta)$  which means that  $y^* \in S(B_{Y^*}, y, \delta)$ .

(iii)  $\Rightarrow$  (ii) If  $A(G, S, \varepsilon)$  is weak\*-dense in  $K(Y^*)$  then by Remark 1 for every  $y \in S_Y$  the  $w^*$ -slice  $S(B_{Y^*}, y, \varepsilon)$  intersects  $A(G, S, \varepsilon)$ . Therefore there is a  $y^* \in A(G, S, \varepsilon)$  such that  $y \in S'(y^*, \varepsilon)$ .

Since  $A(G, S_n, \varepsilon)$  are weak\*-dense and weak\*-open then the equivalence (iii)  $\Leftrightarrow$  (iv) follows from the Baire theorem.  $\square$

**Proposition 5** *Let  $G \in S_{L(X, Y)}$  be a Daugavet center and  $T: X \rightarrow E$  be an SCD-operator. Then  $T$  is a  $G$ -strong Daugavet operator.*

*Proof.* Let  $T$  be an SCD-operator. Then there exists a sequence  $\{S_n: n \in \mathbb{N}\}$  of slices of  $T(B_X)$  such that  $T(B_X) \subseteq \overline{\text{conv}}(B)$  whenever  $B \subseteq T(B_X)$  intersects all the  $S_n$ 's. Remark that the sets  $\hat{S}_n := T^{-1}(S_n) \cap B_X$  are slices of  $B_X$ .

Pick an  $\varepsilon > 0$ , an  $x \in S_X$  and a  $y \in S_Y$ . Since  $G$  is a Daugavet center, item (iv) of Lemma 6 gives us that  $\bigcap_{n \in \mathbb{N}} A(G, \hat{S}_n, \varepsilon/2)$  is weak\*-dense in  $K(Y^*)$ . Remark 1 implies that there is a  $y^* \in \bigcap_{n \in \mathbb{N}} A(G, \hat{S}_n, \varepsilon/2)$  such that

$$y \in S'(y^*, \varepsilon/2). \quad (3)$$

By the definition of  $A(G, \hat{S}_n, \varepsilon/2)$  we have that  $\overline{S'(G^*y^*, \varepsilon/2)} \cap T^{-1}(S_n) \neq \emptyset$  for every  $n \in \mathbb{N}$ . Consequently,

$$T(\overline{S'(G^*y^*, \varepsilon/2)}) \cap S_n \neq \emptyset$$

for every  $n \in \mathbb{N}$ . Then

$$T(B_X) \subseteq \overline{\text{conv}}(T(\overline{S'(G^*y^*, \varepsilon/2)})) = \overline{T(S'(G^*y^*, \varepsilon/2))}.$$

Hence  $Tx \in \overline{T(S'(G^*y^*, \varepsilon/2))}$  which implies that there is a  $z \in S'(G^*y^*, \varepsilon/2)$  such that

$$\|Tx - Tz\| < \varepsilon.$$

We have  $y^*(Gz) > 1 - \varepsilon/2$ . By (3) we also have  $y^*(y) > 1 - \varepsilon/2$ . Therefore

$$\|y + Gz\| \geq \|y^*\| \|y + Gz\| \geq |y^*(y) + y^*(Gz)| > 1 - \varepsilon/2 + 1 - \varepsilon/2 = 2 - \varepsilon.$$

Then  $T$  is a  $G$ -strong Daugavet operator by Definition 4.  $\square$

Using Lemma 1 we obtain the following corollary:

**Corollary 1** *Let  $G \in S_{L(X, Y)}$  be a Daugavet center. If  $T: X \rightarrow Y$  is an SCD-operator then  $\|G + T\| = 1 + \|T\|$ .*

It was shown in [6] that if  $T: X \rightarrow Y$  is a hereditary SCD-operator then for every  $x^* \in X^*$  the operator  $T \upharpoonright x^*$  is an SCD-operator.

**Corollary 2** *Let  $G \in S_{L(X, Y)}$  be a Daugavet center,  $E$  be a Banach space, and let  $T: X \rightarrow E$  be a hereditary SCD-operator. Then  $T$  is  $G$ -narrow.*

**Corollary 3** For a Daugavet center  $G \in S_{L(X,Y)}$  all weakly compact operators, operators not fixing a copy of  $\ell_1$ , and all strong Radon-Nikodým operators on  $X$  are  $G$ -narrow.

**4. Examples of Daugavet centers and  $G$ -narrow operators**

The results of this section are concentrated around the following (in general still open) question: what one can say about two Daugavet centers  $G_1, G_2$  on  $X$ , if  $\mathcal{NAR}(G_1, X) = \mathcal{NAR}(G_2, X)$ ?

**Proposition 6** Let  $G \in S_{L(X,Y)}$  be a Daugavet center and  $V: Y \rightarrow E$  be a surjective isometry. Then  $\mathcal{NAR}(V \circ G, X) = \mathcal{NAR}(G, X)$  and  $\mathcal{SD}(V \circ G, X) = \mathcal{SD}(G, X)$ .

*Proof.* First we prove that  $\mathcal{NAR}(V \circ G, X) = \mathcal{NAR}(G, X)$ . Let  $T \in \mathcal{NAR}(G, X)$ ,  $\varepsilon > 0$ ,  $x \in S_X$ ,  $e \in S_E$  and let  $S$  be a slice of  $B_X$  with  $x \in S$ . Denote  $y := V^{-1}e$ . Remark that  $y \in S_Y$ . Then there is a  $z \in S$  such that  $\|T(x - z)\| < \varepsilon$  and  $\|Gz + y\| > 2 - \varepsilon$ . Hence

$$\|VGz + e\| = \|V(Gz + y)\| = \|Gz + y\| > 2 - \varepsilon.$$

Consequently,  $T$  is a  $V \circ G$ -narrow operator. So we have  $\mathcal{NAR}(G, X) \subset \mathcal{NAR}(VG, X)$ . Since  $V^{-1}$  is also a surjective isometry, then

$$\mathcal{NAR}(V \circ G, X) \subset \mathcal{NAR}(G, X).$$

Thus, we have  $\mathcal{NAR}(V \circ G, X) = \mathcal{NAR}(G, X)$ . One can prove  $\mathcal{SD}(V \circ G, X) = \mathcal{SD}(G, X)$  in the same way. □

**Corollary 4** Let  $X$  have the Daugavet property and  $G: X \rightarrow E$  be a surjective isometry. Then  $\mathcal{NAR}(G, X) = \mathcal{NAR}(X)$  and  $\mathcal{SD}(G, X) = \mathcal{SD}(X)$ .

**Proposition 7** Let  $G \in S_{L(X,Y)}$  be a Daugavet center and  $U: E \rightarrow X$  be a surjective isometry. Then

- (a) For each  $T \in \mathcal{NAR}(G, X)$  the composition  $T \circ U \in \mathcal{NAR}(G \circ U, E)$ .
- (b) For each  $T \in \mathcal{SD}(G, X)$  the composition  $T \circ U \in \mathcal{SD}(G \circ U, E)$ .

*Proof.* We now prove part (a). Let  $\varepsilon > 0$ ,  $e \in S_E$ ,  $y \in S_Y$  and let  $S = S(B_E, x^*, \alpha)$  be a slice with  $x^* \in S_{E^*}$  and  $e \in S$ . Then

$$Ue \in US = \{v \in B_X : x^*(U^{-1}v) > 1 - \alpha\}.$$

Recall that  $\|U^{-1}\| = 1$ , hence  $x^* \circ U^{-1} \in S_{X^*}$  and  $US = S(B_X, x^* \circ U^{-1}, \alpha)$ . Since  $T$  is  $G$ -narrow, there is a  $v_0 \in US$  with  $\|T(Ue - v_0)\| < \varepsilon$  and  $\|Gv_0 + y\| > 2 - \varepsilon$ . Then for  $z := U^{-1}v_0$  we have  $z \in S$ ,  $\|TU(e - z)\| < \varepsilon$  and  $\|GUz + y\| > 2 - \varepsilon$ . This means that  $T \circ U$  is a  $G \circ U$ -narrow operator.

Part (b) can be proved in analogous way. □

**Proposition 8** *Let  $G \in S_{L(X,Y)}$ ,  $J: Y \rightarrow E$  be an isometric embedding and let  $J \circ G$  be a Daugavet center. Then  $G$  is a Daugavet center,  $\mathcal{NAR}(J \circ G, X) \subset \mathcal{NAR}(G, X)$  and  $\mathcal{SD}(J \circ G, X) \subset \mathcal{SD}(G, X)$ .*

*Proof.* Let  $\varepsilon > 0$ ,  $x \in S_X$ ,  $y \in S_Y$  and let  $S$  be a slice of  $B_X$  with  $x \in S$ . Since  $J$  is an isometric embedding then  $Jy \in S_E$ . Let  $T \in \mathcal{NAR}(J \circ G, X)$  then there is a  $z \in S$  such that  $\|T(x - z)\| < \varepsilon$  and

$$\|Gz + y\| = \|J(Gz) + Jy\| > 2 - \varepsilon,$$

which implies that  $T \in \mathcal{NAR}(G, X)$ . Then we have  $\mathcal{NAR}(J \circ G, X) \subset \mathcal{NAR}(G, X)$ . Consequently  $\mathcal{NAR}(G, X) \neq \emptyset$ , and hence  $G$  is a Daugavet center.

The proof of  $\mathcal{SD}(J \circ G, X) \subset \mathcal{SD}(G, X)$  is analogous.  $\square$

Our next goal is to show that there exist  $G$ -narrow operators on  $C(K)$  which are not narrow, and that there are narrow operators on  $C(K)$  which are not  $G$ -narrow for some Daugavet center  $G: C(K) \rightarrow Y$ .

Let  $K \subset [0, 1]$  be a compact set. Consider the restriction operator  $G: C[0, 1] \rightarrow C(K)$  which maps every function  $f$  into its restriction to  $K$ . Theorem 3.7 of [2] implies that  $G$  is a Daugavet center if  $K$  has no isolated points. We use the idea of Theorem 3.7 of [2] to prove the following proposition.

**Proposition 9** *Let  $K_1 \subset [0, 1]$  and  $K_2 \subset [0, 1]$  be compact sets with  $K_1 \cap K_2 = \emptyset$  and let  $K_2$  have no isolated points. Consider  $T: C[0, 1] \rightarrow C(K_1)$ ,  $Tf = f|_{K_1}$  and  $G: C[0, 1] \rightarrow C(K_2)$ ,  $Gf = f|_{K_2}$ . Then  $T$  is a  $G$ -narrow operator.*

*Proof.* Pick an  $\varepsilon > 0$ , an  $x \in S_{C[0,1]}$ , a  $y \in S_{C(K_2)}$  and a slice  $S = S(B_{C[0,1]}, x^*, \alpha)$  which contains  $x$ . We need to find a  $z \in S$  satisfying  $\|Tx - Tz\| < \varepsilon$  and  $\|Gz + y\| > 2 - \varepsilon$ . By the Riesz representation theorem for  $x^*$  there is a unique Borel regular signed measure  $\sigma$  on  $\Delta := [0, 1]$  such that

$$x^*(f) = \int_{\Delta} f d\sigma$$

for all  $f \in C(\Delta)$ , and  $\|x^*\| = |\sigma|(\Delta)$ . So

$$\begin{aligned} S &= \{f \in B_{C(\Delta)}: \int_{\Delta} f d\sigma > |\sigma|(\Delta) - \alpha\} \\ &= \{f \in B_{C(\Delta)}: \int_{\Delta} (1 - (\mathbf{1}_{\Delta^+} - \mathbf{1}_{\Delta^-})f) d|\sigma| < \alpha\}, \end{aligned}$$

where  $\Delta = \Delta^+ \sqcup \Delta^-$  is a Hahn decomposition of  $\Delta$  for  $\sigma$  and  $\mathbf{1}_A$  denotes a characteristic function of the set  $A$ . Since  $x \in S$  then there is an  $\varepsilon_1 > 0$  such that

$$\int_{\Delta} (1 - (\mathbf{1}_{\Delta^+} - \mathbf{1}_{\Delta^-})x) d|\sigma| < \alpha - \varepsilon_1.$$

Let  $t_0 \in K_2$  be a point such that  $|y(t_0)| = 1$ . Without loss of generality we assume that  $y(t_0) = 1$ . Consider a neighborhood  $U \subset K_2$  of a point  $t_0$  such that  $y(t) > 1 - \varepsilon$  for every  $t \in U$ . Since  $K_2$  has no isolated points and  $\sigma$  has at most countable set of atoms then there are  $t_1 \in U$  which is not an atom of  $\sigma$ , and an open neighborhood  $V \subset \Delta$  of  $t_1$  such that  $K_1 \cap V = \emptyset$  and  $|\sigma|(V) < \varepsilon_1/2$ .

Now we construct the needed  $z$ . Put  $z(t_1) = 1$  and  $z = x$  on  $\Delta \setminus V$ . Since  $(\Delta \setminus V) \cup \{t_1\}$  is a closed set then by the Tietze extension theorem we construct a continuous extension of  $z$  on  $V \setminus \{t_1\}$  keeping the condition  $\|z\| = 1$ . Let us show that  $z \in S$ .

$$\begin{aligned} \int_{\Delta} (1 - (\mathbf{1}_{\Delta^+} - \mathbf{1}_{\Delta^-})z) d|\sigma| &= \int_{\Delta \setminus V} (1 - (\mathbf{1}_{\Delta^+} - \mathbf{1}_{\Delta^-})x) d|\sigma| \\ &+ \int_V (1 - (\mathbf{1}_{\Delta^+} - \mathbf{1}_{\Delta^-})z) d|\sigma| \\ &< \int_{\Delta} (1 - (\mathbf{1}_{\Delta^+} - \mathbf{1}_{\Delta^-})x) d|\sigma| + \varepsilon_1 \\ &< \alpha - \varepsilon_1 + \varepsilon_1 = \alpha. \end{aligned}$$

In addition, we have

$$\|Gz + y\| = \sup_{t \in K_2} |z + y| \geq |z(t_1) + y(t_1)| > 2 - \varepsilon$$

and since  $K_1 \cap V = \emptyset$  and  $z = x$  on  $\Delta \setminus V$ , then

$$\|Tx - Tz\| = \sup_{t \in K_1} |x - z| = 0.$$

So,  $T$  is a  $G$ -narrow operator. □

It was proved in [7] that for a compact  $K$  without isolated points and for  $T \in L(C(K), Y)$  the following assertions are equivalent:

- (i)  $T \in \mathcal{NAR}(C(K))$ .
- (ii) For every  $\varepsilon > 0$  and every proper closed subset  $F \subset K$  there is an non-negative  $f \in S_{C(K)}$  with  $f|_F = 0$  and  $\|Tf\| < \varepsilon$ .

**Proposition 10** *Let  $K_0 \subset [0, 1]$  be a compact set. For  $G: C[0, 1] \rightarrow C(K_0)$ ,  $Gf = f|_{K_0}$  the following assertions are equivalent:*

- (i)  $G \in \mathcal{NAR}(C[0, 1])$ .
- (ii)  $K_0$  is nowhere dense in  $[0, 1]$ .

*Proof.* (ii)  $\Rightarrow$  (i). Pick  $\varepsilon > 0$  and  $(a, b) \subset [0, 1]$ . Since  $K_0$  is nowhere dense in  $[0, 1]$  then there is an open set  $U \subset (a, b)$  such that  $U \cap K_0 = \emptyset$ . Then for every non-negative  $f \in S_{C[0,1]}$  with  $\text{supp } f \subset U$  we have  $\|Gf\| = 0$ . Therefore,  $G$  is a narrow operator.

(i)  $\Rightarrow$  (ii). Assume to the contrary that  $K_0$  is dense in some  $(a, b) \subset [0, 1]$ . Consider a non-negative  $f \in S_{C[0,1]}$  with  $\text{supp } f \subset (a, b)$ . Let  $\varepsilon > 0$  then there is a  $t \in K_0$  such that  $|f(t)| > 1 - \varepsilon$ . Hence  $\|Gf\| > 1 - \varepsilon$ . So  $G$  is not narrow. □

**Example 1** Let  $K$  be the Cantor set on  $[0, 1]$  and  $G: C[0, 1] \rightarrow C(K)$ ,  $Gf = f|_K$ . Since Cantor set is nowhere dense then  $G$  is a narrow operator. But Cantor set also has no isolated points, so  $G$  is a Daugavet center and hence is not  $G$ -narrow.

**Example 2** Consider compact sets  $K_1 \subset [0, 1]$  and  $K_2 \subset [0, 1]$  with  $K_1 \cap K_2 = \emptyset$ . Let  $K_1$  contain some open set  $U \subset [0, 1]$  and let  $K_2$  have no isolated points. Consider the restriction operators  $T: C[0, 1] \rightarrow C(K_1)$  and  $G: C[0, 1] \rightarrow C(K_2)$ . Then by Proposition 9  $T$  is  $G$ -narrow, and Proposition 10 implies that  $T$  is not narrow.

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