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Some generalizations of p-loxodromic functions

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The functional equation of the form $f(qz) = p(z)f(z)$, $q \in \mathbb{C}\setminus\{0\}$, $|q| < 1$, $z \in \mathbb{C}\setminus\{0\}$ is considered. For certain fixed elementary functions $p(z)$, meromorphic solutions of this equation are found. These solutions are some generalizations of p-loxodromic functions and can be represented via the Schottky-Klein prime function as well as classic p-loxodromic functions.
Keywords: loxodromic function; p-loxodromic function; the Schottky-Klein prime function.

Христіянин А.Я., Луківська Дз.В. **Деякі узагальнення p-локсодромних функцій.** Розглянуто функціональне рівняння $f(qz) = p(z)f(z)$, $z \in \mathbb{C}\setminus\{0\}$, $q \in \mathbb{C}\setminus\{0\}$, $|q| < 1$. При певних фіксованих елементарних функціях $p(z)$ знайдено його мероморфні розв'язки. Ці розв'язки є деякими узагальненнями p-локсодромних функцій і можуть зображатися за допомогою первинної функції Шотткі-Кляйна, як і класичні p-локсодромні функції.
Ключові слова: локсодромна функція; p-локсодромна функція; первинна функція Шотткі-Кляйна.

Хрыстиянын А.Я., Лукивська Дз.В. **Некоторые обобщения p-локсодромических функций.** Рассмотрено функциональное уравнение $f(qz) = p(z)f(z)$, $z \in \mathbb{C}\setminus\{0\}$, $q \in \mathbb{C}\setminus\{0\}$, $|q| < 1$. При определенных фиксированных элементарных функциях $p(z)$ найдены его мероморфные решения. Эти решения являются некоторыми обобщениями p-локсодромических функций и могут изображаться с помощью первичной функции Шоттки-Кляйна, как и классические p-локсодромические функции.
Ключевые слова: локсодромическая функция; p-локсодромическая функция; первичная функция Шоттки-Кляйна.
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1. Introduction

Denote $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. For $z \in \mathbb{C}^*$ consider the equation of the form

$$f(qz) = p(z)f(z), \tag{1}$$

where $p(z)$ is some function, $q \in \mathbb{C}^*$, $|q| < 1$. If $p(z) \equiv const$, then meromorphic solution of this equation is called p-loxodromic function [5]. In particular, if $p(z) \equiv 1$, we have classic loxodromic function. The class of loxodromic functions is denoted by $\mathcal{L}_q$. It was studied in the works of O. Rausenberger [12], G. Valiron [14] and Y. Hellegouarch [3]. In recent years, A. Kondratyuk and his colleagues also investigated these functions and their various generalizations in other domains (see, for example [4], [6]-[8]).

Loxodromic functions have been used to construct explicit solutions to the rotating Hele-Shaw problem, the viscous sintering problem, the problem of finding vortical equilibria of the Euler equation and the problem of free surface Euler flows of the surface tension [2]. These functions also have a fairly wide range of practical applications, for example see [10], [11].

So, it will be quite interesting to generalize the class of p-loxodromic functions for the case of more general functions $p(z)$ other then the constant ones. The purpose of this article is to obtain meromorphic solutions of the equation (1), where $p(z)$ are some elementary functions. These solutions will be some generalizations of p-loxodromic functions. This task can be viewed as the first step towards more general case where $p(z)$ is an arbitrary rational function, which in turn may lead to further generalizations.

2. The case $p(z) = \frac{1}{z}$

Let us consider functional equation

$$f(qz) = \frac{1}{z}f(z),\ z \in \mathbb{C}^*. \tag{2}$$

Our task is to find its meromorphic in $\mathbb{C}^*$ solutions. At first consider the Schottky-Klein prime function [5]

$$P(z) = (1-z)\prod_{n=1}^{\infty}(1-q^n z)\left(1-\frac{q^n}{z}\right). \tag{3}$$

It was introduced by Schottky [13] and Klein [9] for the study of conformal mappings of double-connected domains, see also [1]. This function is holomorphic in $\mathbb{C}^*$ and has zero sequence $\{q^n\}$, $n \in \mathbb{Z}$. The following property of $P(z)$ is well known [3, p. 94]

$$P(qz) = -z^{-1}P(z). \tag{4}$$

Theorem 1 *Let $g \in \mathcal{L}_q$. The meromorphic in $\mathbb{C}^*$ function $f(z) = P(-z)g(z)$ satisfies (2).*

Proof. The proof is by direct calculation. Since g is loxodromic, we have

$$f(qz) = P(-qz)g(qz) = \frac{1}{z}P(-z)g(z) = \frac{1}{z}f(z).$$

We also use here equality (4).

Theorem 2 *Every meromorphic in $\mathbb{C}^*$ solution of (2) can be represented in the form $f(z) = P(-z)g(z)$, where $g \in \mathcal{L}_q$.*

Proof. Let $f(z)$ be a solution of (2). Consider the function $g(z) = \dfrac{f(z)}{P(-z)}$. Since $f(z)$ is meromorphic and $P(-z)$ is holomorphic, it follows that g is meromorphic. Applying equalities (2) and (4), we get

$$g(qz) = \frac{f(qz)}{P(-qz)} = \frac{\frac{1}{z}f(z)}{\frac{1}{z}P(-z)} = g(z).$$

Therefore, for all $z \neq -q^n$, $n \in \mathbb{Z}$ we have $g(qz) = g(z)$. It means that g is loxodromic, which concludes the proof.

We also can reformulate Theorems 1 and 2 in the following forms.

Theorem 3 *The meromorphic in $\mathbb{C}^*$ function*

$$f(z) = C\frac{P\left(\frac{z}{a_1}\right)P\left(\frac{z}{a_2}\right)\dots P\left(\frac{z}{a_m}\right)P\left(\frac{z}{a_{m+1}}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\dots P\left(\frac{z}{b_m}\right)},$$

where C is a constant, $a_1, a_2, \dots, a_{m+1}$ and $b_1, b_2, \dots, b_m$ are complex numbers, not necessarily distinct, such that $\prod_{j=1}^{m+1} a_j = -\prod_{j=1}^{m} b_j$, satisfies equation (2).

Proof. Indeed, taking into account equality (4),

$$f(qz) = C\frac{P\left(\frac{qz}{a_1}\right)P\left(\frac{qz}{a_2}\right)\dots P\left(\frac{qz}{a_m}\right)P\left(\frac{qz}{a_{m+1}}\right)}{P\left(\frac{qz}{b_1}\right)P\left(\frac{qz}{b_2}\right)\dots P\left(\frac{qz}{b_m}\right)}$$

$$=C\frac{-\frac{a_1}{z}P\left(\frac{z}{a_1}\right)\left(-\frac{a_2}{z}\right)P\left(\frac{z}{a_2}\right)\dots\left(-\frac{a_m}{z}\right)P\left(\frac{z}{a_m}\right)\left(-\frac{a_{m+1}}{z}\right)P\left(\frac{z}{a_{m+1}}\right)}{-\frac{b_1}{z}P\left(\frac{z}{b_1}\right)\left(-\frac{b_2}{z}\right)P\left(\frac{z}{b_2}\right)\dots\left(-\frac{b_m}{z}\right)P\left(\frac{z}{b_m}\right)}$$

$$=C\frac{(-1)^{m+1}a_1a_2\dots a_{m+1}}{(-1)^m b_1b_2\dots b_m}\frac{P\left(\frac{z}{a_1}\right)P\left(\frac{z}{a_2}\right)\dots P\left(\frac{z}{a_m}\right)P\left(\frac{z}{a_{m+1}}\right)\left(\frac{1}{z}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\dots P\left(\frac{z}{b_m}\right)}=\frac{1}{z}f(z)$$

Theorem 4 *Every meromorphic in $\mathbb{C}^*$ solution of equation (2) can be written in the form*

$$f(z) = C\frac{P\left(\frac{z}{a_1}\right)P\left(\frac{z}{a_2}\right)\dots P\left(\frac{z}{a_m}\right)P\left(\frac{z}{a_{m+1}}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\dots P\left(\frac{z}{b_m}\right)},$$

where C is a constant, $a_1, a_2, \dots, a_{m+1}$ and $b_1, b_2, \dots, b_m$ are complex numbers, not necessarily distinct, such that $\prod_{j=1}^{m+1} a_j = -\prod_{j=1}^{m} b_j$.

Proof. By Theorem 2 we know that

$$f(z) = P(-z)g(z), \tag{5}$$

where $g \in \mathcal{L}_q$. We use the loxodromic function representation via Schottky-Klein prime functions (see [3], [14] for more details). Namely, let $c_1, c_2, ..., c_m$ and $b_1, b_2 ..., b_m$ be the zeros and the poles of function g in the annulus $A_q(R) = \{z \in \mathbb{C} : |q|R < |z| \le R\}$, $R > 0$, respectively, $\partial A_q(R)$ contains neither zeros nor poles of $g \in \mathcal{L}_q$. Note that each loxodromic function g has equal numbers of zeros and poles (counted according to their multiplicities) in every such annulus $A_q(R)$ [3, p. 93]. Then [14, p. 478]

$$g(z) = Kz^p\frac{P\left(\frac{z}{c_1}\right)P\left(\frac{z}{c_2}\right)\cdot ... \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\cdot ... \cdot P\left(\frac{z}{b_m}\right)}, \tag{6}$$

where

$$\frac{c_1 c_2 \dots c_m}{b_1 b_2 \dots b_m} = q^{-p}, \quad p \in \mathbb{Z}, \tag{7}$$

and K is a constant. Applying equality (4) to (6), we have

$$g(z) = C\frac{P\left(\frac{z}{q^p c_1}\right)P\left(\frac{z}{c_2}\right)\cdot ... \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\cdot ... \cdot P\left(\frac{z}{b_m}\right)}. \tag{8}$$

where $C = (-a_1)^p q^{\frac{p(p+1)}{2}} K$. Combining (5) and (8), we obtain

$$f(z) = C\frac{P\left(\frac{z}{q^p c_1}\right)P\left(\frac{z}{c_2}\right)\dots P\left(\frac{z}{c_m}\right)P\left(\frac{z}{-1}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\dots P\left(\frac{z}{b_m}\right)},$$

Let us denote $a_1 = q^p c_1$, $a_2 = c_2, \ldots, a_m = c_m$, $a_{m+1} = -1$. Now we can rewrite f as follows

$$f(z) = C\frac{P\left(\frac{z}{a_1}\right)P\left(\frac{z}{a_2}\right)\ldots P\left(\frac{z}{a_m}\right)P\left(\frac{z}{a_{m+1}}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\ldots P\left(\frac{z}{b_m}\right)},$$

where $\prod\limits_{j=1}^{m+1} a_j = -\prod\limits_{j=1}^{m} b_j$, which is clear in view of (7). The theorem is proved.

3. The case $p(z) = \dfrac{1}{1-z}$

Now, consider functional equation of the form

$$f(qz) = \frac{1}{1-z} f(z),\ z \in \mathbb{C}^*. \tag{9}$$

We also are interested in finding meromorphic in $\mathbb{C}^*$ solutions of (9).

Define the entire function with the zero sequence $\{q^{-n}\}$, $n \in \mathbb{N} \cup \{0\}$, $0 < |q| < 1$,

$$H(z) = \prod_{n=0}^{\infty}(1 - q^n z).$$

Theorem 5 *Let $g \in \mathcal{L}_q$. The meromorphic in $\mathbb{C}^*$ function $f(z) = H(z)g(z)$ satisfies (9).*

Proof. The proof is straightforward. Since g is loxodromic, we have

$$(1-z)f(qz) = (1-z)g(qz)H(qz) = (1-z)g(z)\prod_{n=0}^{\infty}(1-q^{n+1}z)$$

$$= (1-z)g(z)\prod_{k=1}^{\infty}(1-q^k z) = g(z)\prod_{n=0}^{\infty}(1-q^n z) = f(z).$$

Theorem 6 *Every meromorphic in $\mathbb{C}^*$ solution of (9) can be represented in the form $f(z) = H(z)g(z)$, where $g \in \mathcal{L}_q$.*

Proof. The proof is analogous to the proof of Theorem 2. Let f be a solution of equation (9). Consider the function $g = \dfrac{f}{H}$. Since f is meromorphic and H is holomorphic, it follows that g is meromorphic. Taking into account equality (9), we get

$$g(qz) = \frac{f(qz)}{H(qz)} = \frac{\frac{1}{1-z}f(z)}{\frac{1}{1-z}H(z)} = g(z).$$

Therefore, for all $z \neq q^{-n}$, $n \in \mathbb{N} \cup \{0\}$ we can conclude that $g(qz) = g(z)$. We obtain that g is loxodromic. The proof is completed.

Using the loxodromic function representation via Schottky-Klein prime functions, namely formulas (6) and (7), we also can rewrite Theorems 5 and 6 in the following forms.

Theorem 7 *The meromorphic in* $\mathbb{C}^*$ *function*

$$f(z) = Cz^p H(z) \frac{P\left(\frac{z}{c_1}\right) P\left(\frac{z}{c_2}\right) \cdot \ldots \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot \ldots \cdot P\left(\frac{z}{b_m}\right)},$$

where $c_1, c_2 \ldots, c_m$ *and* $b_1, b_2 \ldots, b_m$ *are complex numbers, not necessarily distinct, such that* $q^p \prod\limits_{j=1}^{m} c_j = \prod\limits_{j=1}^{m} b_j$, $p \in \mathbb{Z}$ *and* C *is a constant, satisfies (9).*

Theorem 8 *Every meromorphic in* $\mathbb{C}^*$ *solution of (9) can be written in the form*

$$f(z) = Cz^p H(z) \frac{P\left(\frac{z}{c_1}\right) P\left(\frac{z}{c_2}\right) \cdot \ldots \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot \ldots \cdot P\left(\frac{z}{b_m}\right)},$$

where $c_1, c_2 \ldots, c_m$ *and* $b_1, b_2 \ldots, b_m$ *are complex numbers, not necessarily distinct, such that* $q^p \prod\limits_{j=1}^{m} c_j = \prod\limits_{j=1}^{m} b_j$, $p \in \mathbb{Z}$ *and* C *is a constant.*

Applying the Schottky-Klein prime function's property (4) to the representation of function f in Theorems 7 and 8 we can reformulate these theorems in the next forms.

Theorem 9 *The meromorphic in* $\mathbb{C}^*$ *function*

$$f(z) = CH(z) \frac{P\left(\frac{z}{q^p c_1}\right) P\left(\frac{z}{c_2}\right) \cdot \ldots \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot \ldots \cdot P\left(\frac{z}{b_m}\right)},$$

where $c_1, c_2 \ldots, c_m$ *and* $b_1, b_2 \ldots, b_m$ *are complex numbers, not necessarily distinct, such that* $q^p \prod\limits_{j=1}^{m} c_j = \prod\limits_{j=1}^{m} b_j$, $p \in \mathbb{Z}$ *and* C *is a constant, satisfies (9).*

Theorem 10 *Every meromorphic in* $\mathbb{C}^*$ *solution of (9) can be represented in the form*

$$f(z) = CH(z) \frac{P\left(\frac{z}{q^p c_1}\right) P\left(\frac{z}{c_2}\right) \cdot \ldots \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot \ldots \cdot P\left(\frac{z}{b_m}\right)},$$

where $c_1, c_2 \ldots, c_m$ *and* $b_1, b_2 \ldots, b_m$ *are complex numbers, not necessarily distinct, such that* $q^p \prod_{j=1}^{m} c_j = \prod_{j=1}^{m} b_j$, $p \in \mathbb{Z}$ *and* C *is a constant.*

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