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# The convergence of sequences of canonical potentials in the space $L_{1,loc}(\mathbb{C})$

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Potential theory is important in the theory of subharmonic and $\delta$-subharmonic functions. In the article we sharpen Azarin's variant on the convergence of the sequence of canonical potentials in the space $L_{1,loc}(\mathbb{C})$.
*Keywords:* canonical potential, Radon measure, widely convergence.

Нгуен Ван Куинь. **Сходимость последовательности канонических потенциалов в пространстве $L_{1,loc}(\mathbb{C})$.** В теории субгармонических и $\delta$-субгармонических функций существенную роль играет теория потенциала. В статье предлагается усиление варианта Азарина теоремы о сходимости последовательности канонических потенциалов в пространстве $L_{1,loc}(\mathbb{C})$.
*Ключевые слова:* канонический потенциал, мера Радона, широкая сходимость.

Нгуєн Ван Куїнь. **Збіжність послідовності канонічних потенціалів в просторі $L_{1,loc}(\mathbb{C})$.** У теорії субгармонічних і $\delta$-субгармонічних функцій суттєву роль відіграє теорія потенціалу. У статті пропонується посилення варіанту Азаріна теореми про збіжність послідовності канонічних потенціалів в просторі $L_{1,loc}(\mathbb{C})$.
*Ключові слова:* канонічний потенціал, міра Радону, широка збіжність.

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The study of the potential theory and related problems in mathematical physics has been in the focus of mathematicians since the nineteenth century. In particular, in the study of subharmonic and $\delta$ - subharmonic functions, methods of the potential theory play an important role. The results in the present paper can be viewed as the versions of some theorems from monographs of N.S.Landkof [2] and V.S. Azarin [1]. See also the paper of A. F. Grishin, N. Quynh, and I. Podiedtseva [3], where the representation theorem for $\delta$ -subharmonic functions of finite order in the form of canonical potentials was proved, and the paper of A. F. Grishin and A. Shuigi [4], in which various types of convergence of sequences of $\delta$-subharmonic functions were studied. The results of our article allow us to simplify to some extent the constructions from these articles.

In Section 1 we give the necessary definitions and known results in convenient formulations, where we follow [2] and [1]. In the main section 2 we give new theorems on the convergence of the sequence of canonical potentials. Note that when studying canonical potentials, it is necessary to evaluate separately the corresponding integrals for $\zeta| < |z|$ and $|\zeta| > |z|$, since the kernels in these cases are different. Therefore, these cases are considered separately.

## 1. Preliminary results

We will use the following notation:

$$B(0, R) = \{z \in \mathbb{C} : \ |z| \le R\};$$
$$C(0, R) = \{z \in \mathbb{C} : \ |z| < R\};$$
$$S(0, R) = \{z \in \mathbb{C} : \ |z| = R\};$$
$$R([R_1, R_2]) = B(0; R_2)/C(0, R_1).$$

A proximate order is an important tool for investigating the functions of finite order.

An absolutely continuous function $\rho(r)$ on the semiaxis $(0, \infty)$ is called a *proximate order* ( in the sense of Valiron [5]), if two conditions hold: 1) there exists the limit $\rho = \lim\limits_{r\to\infty} \rho(r)$,
2) $\lim\limits_{r\to\infty} r\rho'(r)\ln r = 0$ (under $\rho'(r)$ we mean the maximum modulus of the number of derivative).

In the case when $\rho = 0$, the proximate order $\rho(r)$ is called the zero proximate order. We denote $V(r) = r^{\rho(r)}$. The proximate order $\rho(r)$ is called a *proximate order of the function* $f$ if

$$\sigma = \varlimsup_{r\to\infty} \frac{f(r)}{V(r)} \in (0, \infty). \tag{1}$$

By the equality (1), the value of $\sigma$ is defined for an arbitrary positive function $f$ and an arbitrary proximate order $\rho(r)$. It is called the type of the function $f$ with respect to the proximate order $\rho(r)$. In general, $\sigma \in [0, \infty]$. If the inequality $\sigma < \infty$ holds, then $f(r)$ is called a function of no higher than normal type with respect to the proximate order $\rho(r)$.

In the case of an arbitrary refined order, an additional condition on the proximate order looks as follows. An arbitrary proximate order $\rho(r)$ is represented in the form $\rho(r) = \rho + \rho_1(r)$, where $\rho_1(r)$ is the zero proximate order.

The properties of proximate orders can be found in [7], [6], [8]. Let us formulate several of them we need in what follows.

**Theorem 1** *(See [6], Chapter 1, § 12, Lemma 5). Let $\rho(r)$ be an arbitrary proximate order. Then for any $t > 0$*

$$\lim_{r\to\infty} \frac{V(rt)}{V(r)} = t^{\rho},$$

*and there is a uniform convergence on any segment $[a, b] \subset (0, \infty)$.*

**Theorem 2** *(See for example [8], Theorem 2.5). Let $\rho(r)$ be a zero proximate order. Let*

$$\gamma(t) = \sup_{r>0} \frac{V(rt)}{V(r)}.$$

*Then $\gamma(t)$ – $\gamma(t)$ is a continuous function on the semiaxis $(0, \infty)$, moreover the functions $\gamma(t)$ and $\gamma(\frac{1}{t})$ have zero order, that is*

$$\lim_{t\to\infty} \frac{\ln \gamma(t)}{\ln t} = \lim_{t\to\infty} \frac{\ln \gamma(\frac{1}{t})}{\ln t} = 0.$$

**Remark.** There is a global inequality

$$V(rt) \leq \gamma(t)V(r), \; r, t > 0, \tag{2}$$

where $\rho(r)$ is the zero proximate order. If $\rho(r)$ is an arbitrary proximate order, then

$$V(rt) = (rt)^{\rho(rt)} = t^{\rho} r^{\rho} (rt)^{\rho_1(rt)} \leq \gamma(t) t^{\rho} r^{\rho} V_1(r) = \gamma(t) t^{\rho} V(r), \tag{3}$$

where $\rho = \rho(\infty)$, and the function $\gamma(t)$ is constructed using the zero proximate order $\rho_1(r)$.

We define a Radon measure as the difference of two locally finite Borel measures $\mu = \mu_1 - \mu_2$. If $\mu = \mu_1 - \mu_2$ is such a representation, then the measure $\mu_1$ is called a positive part of the Radon measure $\mu$ and is denoted by $\mu^+$. The measure $\mu_2$ is called a negative part of $\mu$ and is denoted by $\mu^-$. The measure $|\mu| = \mu^+ + \mu^-$ is the modulus of the measure $\mu$.

For Radon measures $\mu$ the domain of definition consists of all Borel sets $E \subset G \subset \mathbb{C}$ except for those $E$ for which $\mu_1(E) = \mu_2(E) = +\infty$.

If there exists a Radon measure $\mu$ such that for any continuous compactly supported function $\varphi$ the relation

$$\lim_{n\to\infty} \int \varphi(x) d\mu_n(x) \to \int \varphi(z) d\mu_n(x)$$

holds, we say that the sequence $\mu_n$ widely converges to $\mu$.

Let $\mu$ be the Radon measure in $\mathbb{C}$, $\rho(r)$ the proximate order. The value

$$\sigma = \lim_{r\to\infty} \frac{|\mu|(B(0, r))}{V(r)}$$

is called a type of $\mu$ with respect to the proximate order $\rho(r)$.

If $\sigma < \infty$, then the measure $\mu$ is called a measure of no higher than normal type with respect to the proximate order $\rho(r)$.

If the measure $\mu$ is such a measure, then there exists a constant $C$ such that for $r \geq 1$ we have the inequality

$$|\mu|(B(0,r)) \leq CV(r). \tag{4}$$

If the measure $\mu$ does not load the disk $B(0,1)$, then the inequality (4) holds for all $r > 0$.

Given $\mu$ and $\rho(r)$ as above, we denote by $\mu_t$ $(t > 0)$ the following measure

$$\mu_t(E) = \frac{\mu(tE)}{V(t)}.$$

The set of measures $\nu = \lim\limits_{n\ to\infty} \mu_{t_n}$, where $t_n \to \infty$, is called the Azarin limit set of the measure $\mu$ ( with respect to the proximate order $\rho(r)$) and is denoted by $Fr[\mu]$ or if $Fr[\mu, \rho(r)]$, should the need arise.

The results below follow in essence from previous definitions and statements.

**Theorem 3** *Let $\mu$ be the Radon measure in $\mathbb{C}$ of no higher than normal type with respect to the proximate order $\rho(r)$, which does not load the disk $B(0,1)$. Then there exists a constant $C$ such that for $t > 0$ and $r > 0$ the inequality*

$$|\mu_t|(B(0,r)) \leq C\gamma(r)r^\rho, \ \rho = \rho(\infty)$$

*holds.*

*Proof.* Taking into account the inequality (4), we have

$$|\mu_t|(B(0,r)) = \frac{|\mu|(B(0,rt))}{V(t)} \leq C\frac{V(rt)}{V(t)}.$$

The inequality (3) completes the proof.

**Theorem 4** *(See [9], Theorem 1). Let $\mu$ be the Radon measure in $\mathbb{C}$ of the type $\sigma$ with respect to the proximate order $\rho(r)$, $\rho = \rho(\infty) > 0$. Let the measure $\mu_t$ be constructed by using the proximate order $\rho(r)$. Then for any measure $\nu \in Fr[\mu]$ and any $r > 0$ the inequality*

$$|\nu|(B(0,r)) \leq \sigma r^\rho$$

*holds.*

In the theory of subharmonic and $\delta$-subharmonic functions in the plane $\mathbb{C}$, an important role is played by the kernel

$$K_p(z,\zeta) = Re\left(\ln\left(1 - \frac{z}{\zeta}\right) + \frac{z}{\zeta} + \cdots + \frac{1}{p}\frac{z^p}{\zeta^p}\right),$$

where $p \in \mathbb{N}$. For all $z,\ \zeta \in \mathbb{C}$ we have the inequality (see [6], Lemma 2)

$$|K_p(z,\zeta)| \leq M(p)\frac{|z|^p}{|\zeta|^p}\min\left\{1, \frac{|z|}{|\zeta|}\right\}, \tag{5}$$

where $M(p)$ depends only on $p$. Let $\mu$ be the Radon measure in $\mathbb{C}$. We consider the following potential

$$\int\limits_{\mathbb{C}} K_p(z;\zeta)d\mu(\zeta),$$

which we call *the canonical potential of the measure* $\mu$.

The convergence of the sequence $\upsilon_n(z)$ to $\upsilon(z)$ in the space $L_{1,loc}(\mathbb{C})$ means

$$\int |\upsilon_n(z) - \upsilon(z)|d\gamma(z) \to 0 \quad (n \to \infty),$$

where the measure $\gamma$ is the restriction of the Lebesgue measure on the compact set $K \subset \mathbb{C}$.

## 2. Main results

In this section we prove a series of results on the convergence of a sequence of canonical potentials.

**Theorem 5** *Let $\rho(r)$ be a proximate order, $\rho = \rho(\infty) \geq 1$ be an integer. Let the measure $\mu$ be a measure of no higher than normal type with respect to the proximate order $\rho(r)$, which does not load the disk $B(0,1)$. Let the sequence of measures $\mu_{t_n}$ $(t_n \to \infty)$ widely converge to the measure $\nu$. Then the sequence of functions*

$$\upsilon_n(z) = \int\limits_{B(0,|z|)} K_{\rho-1}(z,\zeta)d\mu_{t_n}(\zeta)$$

*converges to a function*

$$\upsilon(z) = \int\limits_{B(0,|z|)} K_{\rho-1}(z,\zeta)d\nu(\zeta)$$

*in the spaces $L_{1,loc}(\mathbb{C})$.*

*Proof.* Since $mu$ is a measure of no higher than normal type with respect to the proximate order $\rho(r)$, which does not load the disk $B(0,1)$, it follows that there exists a constant $M_1$ such that for all $r > 0$ we have the inequality

$$|\mu_{t_n}|(B(0,r)) \leq M_1\frac{V(t_nr)}{V(t_n)}.$$

Applying the inequality (3), we obtain $|\mu_{t_n}|(B(0,r) \le M_1\gamma(r)r^\rho$. Since $\nu \in Fr[\mu]$, it follows from the theorem 4 that there exists a constant $M_2$ such that for all $r > 0$ the inequality $|\nu|(B(0,r)) \le M_2 r^\rho$ holds.

We denote by $\alpha_n = \mu_{t_n} - \nu$. Then there exists a constant $M$ such that we have

$$|\alpha_n|(B(0,r)) \le M_3\gamma(r)r^\rho, \ r > 0. \tag{6}$$

Let $d$ be an arbitrary number with $d \ge 2$. We have

$$\begin{aligned} A_n &= \int\limits_{B(0,d)} |\upsilon_n(z) - \upsilon(z)| dm_2(z) \\ &= \int\limits_{B(0,d)} \left| \int\limits_{B(0,|z|)} K_{\rho-1}(z,\zeta) d\alpha_n(\zeta) \right| dm_2(z) \\ &= \int\limits_{B(0,d)} \int\limits_{B(0,d)} s(z)\chi_{B(0,|z|)}(\zeta) K_{\rho-1}(z,\zeta) d\alpha_n(\zeta) dm_2(z), \end{aligned} \tag{7}$$

where

$$s(z) = sign\ g(z),$$

$$g(z) = \int\limits_{B(0,|z|)} K_{\rho-1}(z,\zeta) d\alpha_n(\zeta) = \sum_{k=0}^{\rho-1} f_k(z),$$

$$f_0(z) = \int\limits_{B(0,|z|)} \ln\left|1 - \frac{1}{\zeta}\right| d\alpha_n(\zeta),$$

$$f_k(z) = \frac{1}{k} Re \left( z^k \int\limits_{B(0,|z|)} \frac{1}{\zeta^k} d\alpha_n(\zeta) \right), \ k = \overline{1, \rho - 1}.$$

The function $\int\limits_{B(0,|z|)} \frac{1}{\zeta^k} d\alpha_n(\zeta)$ is a linear combination with complex coefficients of increasing functions of the variable $|z|$. Therefore, this function is a Borel function in $\mathbb{C}$. From this it follows that the functions $f_k(z), \ k = \overline{1, \rho - 1}$, are Borel functions in the plane $\mathbb{C}$ as well.

Consider the function

$$\tilde{f}_0(z) = \int\limits_{B(0,|z|)} \ln|z-\zeta|\, d\alpha_n(\zeta) = \int\limits_{B(0,|z|)} \chi_{B(0,|z|)}(\zeta) \ln|z-\zeta|\, d\alpha_n(\zeta).$$

The function $\chi_{B(0,|z|)}(\zeta) \ln|z-\zeta|$ is a Borel function of the variables $z, \ \zeta$.

We have

$$I = \int\limits_{B(0,d)} \int\limits_{B(0,d)} \chi_{B(0,|z|)}(\zeta) \left|\ln|z-\zeta|\right| d\alpha_n(\zeta)$$
$$\leq \int\limits_{B(0,d)} \left( \int\limits_0^d \left( \int\limits_0^{2\pi} \left|\ln\left|re^{i\varphi}-\zeta\right|\right| d\varphi \right) rdr \right) d|\alpha_n|(\zeta).$$

Next, we find

$$\begin{aligned}\int\limits_0^{2\pi} \left|\ln|re^{i\varphi}-\zeta|\right| d\varphi &= \int\limits_0^{2\pi} \left(2\ln^+|re^{i\varphi}-\zeta| - 2\ln|re^{i\varphi}-\zeta|\right) d\varphi \\ &\leq 4\pi \ln 2d + 4\pi \min\left(\ln\frac{1}{r},\ \ln\frac{1}{|\zeta|}\right).\end{aligned} \tag{8}$$

$$\begin{aligned}&\int\limits_0^d \left( \int\limits_0^{2\pi} \left|\ln\left|re^{i\varphi}-\zeta\right|\right| d\varphi \right) rdr \\ &\leq 2\pi d^2 \ln 2d - 2\pi \left( 2\ln|\zeta| \int\limits_0^{|\zeta|} rdr + 2\int\limits_{|\zeta|}^d r\ln r dr \right) \leq M_4(d).\end{aligned} \tag{9}$$

From these inequalities it follows that $I$ is finite. This and the Tonelli theorem [10] imply that the function $\chi_{B(0,|z|)} \ln|z-\zeta|$ belongs to the space $L_1(B(0,d)\times B(0,d), dm_2 \times d\alpha_n)$. Next, the Fubini theorem [10] implies that the function $\tilde{f}_0(z)$ is integrable with respect to $m_2$ and, in particular, is a Borel function. We successively obtain that the functions $f_0(z), g(z), s(z)$ are Borel functions and so the function $h(z,\zeta) = s(z)\chi_{B(0,|z|)}(\zeta)K_{\rho-1}(z,\zeta)$ of the variables $z, \zeta$ is also the Borel finction on every set $B(0,d)\times B(0,d)$.

The finiteness of the integral

$$\int\limits_{B(0,d)} \left( \int\limits_{B(0,d)} |h(z,\zeta)| dm_2(z) \right) d|\alpha_n|(\zeta).$$

can be proved in the same way as the finiteness of $I$ above.

Now Tonelli's theorem [10] implies

$$h(z,\zeta) \in L_1(B(0,d)\times B(0,d), dm_2 \times d\alpha_n). \tag{10}$$

Note that this means the finiteness of the four integrals

$$\int\limits_{B(0,d)} \int\limits_{B(0,d)} \left(h(z,\zeta)^{\pm}\right) dm_2(z) d\alpha_n^{\pm}(\zeta).$$

From the equality ( ref q8) and the Fubini theorem  cite K it follows that

$$A_n = \int\limits_{B(0,d)} p(\zeta)d\alpha_n(\zeta), \tag{11}$$

where

$$p(\zeta) = \int\limits_{B(0,d)} s(z)\chi_{B(0,|z|)}(\zeta)K_{\rho-1}(z,\zeta)dm_2(z) = \int\limits_{R([|\zeta|,d])} s(z)K_{\rho-1}(z,\zeta)dm_2(z).$$

Let us prove that the function $p(\zeta)$ is continuous on the set $\mathbb{C}\setminus\{0\}$. It is easy to see that the continuity of $p(\zeta)$ follows from the continuity of the function

$$q(\zeta) = \int\limits_{R([|\zeta|,d])} \ln|z-\zeta|s(z)dm_2(z).$$

We assume for definiteness that the inequality $|\zeta_0| \le |\zeta|$ holds. We have

$$|q(\zeta)-q(\zeta_0)| \le \int\limits_{R([|\zeta_0|,d])} \left|\ln\left|\frac{z-\zeta}{z-\zeta_0}\right|\right| dm_2(z) + \int\limits_{R([|\zeta_0|,|\zeta|])} |\ln|z-z_0||\, dm_2(z)$$

$$\le \int\limits_{R([|\zeta_0|,d])} \ln\left(1+\frac{|\zeta-\zeta_0|}{|z-\zeta_0|}\right) dm_2(z) + \int\limits_{|\zeta_0|}^{|\zeta|}\left(\int\limits_0^{2\pi} |\ln|re^{i\varphi}-\zeta_0||d\varphi\right) rdr$$

$$= J_1 + J_2.$$

The inequality (8) implies the inequality

$$J_2 \le 2\pi|\zeta|\left(\ln 2d + \ln\frac{1}{|\zeta_0|}\right)(|\zeta|-|\zeta_0|).$$

Also, the inequality

$$J_1 \le \int\limits_{B(\zeta_0,2d)} \ln\left(1+\frac{|\zeta-\zeta_0|}{|z-\zeta_0|}\right) dm_2(z)$$

holds.

The integral $J_2$ can be estimated from above by using the polar coordinates with the vertex at $\zeta_0$

$$J_2 \le 2\pi\int\limits_0^{2d} r\ln\left(1+\frac{|\zeta-\zeta_0|}{r}\right) dr.$$

Now the continuity of the function $q(\zeta)$ on the set $\mathbb{C}\setminus\{0\}$ is obvious. Thus, we have proved the continuity of the function $p(\zeta)$ on the set $\mathbb{C}\setminus\{0\}$.

Note that the equality $p(\zeta)=0$ holds for $|\zeta|=d$. If we assume that $p(\zeta)=0$ for $|\ |\geq d$, then the equality (11) can be rewritten as

$$A_n=\int p(\zeta)d\alpha_n(\zeta), \tag{12}$$

where $p(\ zeta)$ is a continuous function compactly supported on the set $\mathbb{C}\setminus\{0\}$. If the function $p(\zeta)$ were continuous in the whole plane, then the equality 12 would already imply the relation $A_n\to 0$ $(n\to\infty)$. However, this is not the case. Therefore, additional reasoning is required. We evaluate the function $p(\zeta)$. The following estimate stems from the inequalities (5), (9)

$$|p(\zeta)|\leq\frac{M_1(d)}{|\zeta|^{\rho-1}},\ |\zeta|\leq 1.$$

Now let $\varepsilon$ be an arbitrary number in the interval $(0,\frac{1}{2})$, $1=\psi_1(\zeta)+\psi_2(\zeta)$ – a continuous partition of unity such that $supp\ \psi_1\subset B(0,2\varepsilon)$, $supp\ \psi_2\cap B(0,\varepsilon)=\emptyset$. Then it follows that

$$\begin{aligned}\overline{\lim_{n\to\infty}}A_n&\leq\overline{\lim_{n\to\infty}}M_1(d)\int\limits_{B(0,2\varepsilon)}\frac{d|\alpha_n|(\zeta)}{|\zeta|^{\rho-1}}+\overline{\lim_{n\to\infty}}\left|\int\psi_2(\zeta)p(\zeta)d\alpha_n(\zeta)\right|\\&=M_2(d)\overline{\lim_{n\to\infty}}\int\limits_{B(0,2\varepsilon)}\frac{d|\alpha_n|(\zeta)}{|\zeta|^{\rho-1}}.\end{aligned}\tag{13}$$

If $\rho=1$, it is easily seen from the resulting inequality that $A_n\to 0$ $(n\to\infty)$. In what follows we assume that $\rho>1$. We have

$$\int\limits_0^{2\varepsilon}\frac{d|\alpha_n|(t)}{t^{\rho-1}}=\frac{|\alpha_n|(B(0,2\varepsilon)}{(2\varepsilon)^{\rho-1}}+\frac{1}{\rho-1}\int\limits_0^{2\varepsilon}\frac{|\alpha_n|(B(0,t))}{t^{\rho}}dt.$$

The latter equality along with the inequalities (6), (13) completes the proof of the theorem.

**Theorem 6** *Let $\rho(r)$ be a proximate order, $\rho=\rho(\infty)\geq 1$ be an integer. Let the measure $\mu$ be a measure of no higher than normal type with respect to the proximate order $\rho(r)$, which does not load the disk $B(0,1)$. Let the sequence of measures $\mu_{t_n}$ $(t_n\to\infty)$ widely converge to the measure $\nu$. Then the sequence of functions*

$$\upsilon_n(z)=\int\limits_{CB(0,|z|)}K_\rho(z,\zeta)d\mu_{t_n}(\zeta)$$

*converges to a function*

$$\upsilon(z)=\int\limits_{CB(0,|z|)}K_\rho(z,\zeta)d\nu(\zeta)$$

*in the spaces $L_{1,loc}(\mathbb{C})$.*

*Proof.* We denote $\alpha_n = \mu_{t_n} - \nu$. Let $d$ be an arbitrary number satisfying $d \geq 2$. We have

$$\begin{aligned} B_n &= \int\limits_{B(0,d)} |v_n(z) - v(z)| dm_2(z) \\ &= \int\limits_{B(0,d)} \left| \int\limits_{CB(0,|z|)} K_\rho(z,\zeta) d\alpha_n(\zeta) \right| dm_2(z) \\ &= \int\limits_{B(0,d)} \int s(z) \chi_{CB(0,|z|)}(\zeta) K_\rho(z,\zeta) d\alpha_n(\zeta) dm_2(z), \end{aligned} \tag{14}$$

where

$$s(z) = sign \int\limits_{CB(0,|z|)} K_\rho(z,\zeta) d\alpha_n(\zeta).$$

Let $N > d$ be an arbitrary number, $1 = \psi_1(\zeta) + \psi_2(\zeta)$ be a continuous partition of unity such that $supp\ \psi_1 \subset B(0,2N)$, $supp\ \psi_2 \cap B(0,N) = \emptyset$. Then the equality (14) can be rewritten in the form

$$\begin{aligned} B_n &= \int\limits_{B(0,d)} \int\limits_{R((|z|,2N])} s(z)\psi_1(\zeta) K_\rho(z,\zeta) d\alpha_n(\zeta) dm_2(z) \\ &+ \int\limits_{B(0,d)} \int\limits_{CB(0,N)} s(z)\psi_2(\zeta) K_\rho(z,\zeta) d\alpha_n(\zeta) dm_2(z) \\ &= J_{1,n} + J_{2,n}. \end{aligned} \tag{15}$$

We investigate each of these integrals. We have

$$J_{1,n} = \int\limits_{B(0,d)} \int\limits_{B(0,2N)} h(z,\zeta) d\alpha_n(\zeta) dm_2(z),$$

where $h(z,\zeta) = s(z)\psi_1(\zeta)\chi_{CB(0,|z|)} K_\rho(z,\zeta)$. Next, repeating the reasoning in the theorem 5, we obtain $h(z,\zeta) \in L_1(B(0,d) \times B(0,2N), dm_2 \times d\alpha_n)$. From this and the Fubini theorem [10] it follows that

$$J_{1,n} = \int\limits_{B(0,2N)} p(\zeta) d\alpha_n(\zeta), \tag{16}$$

$$\begin{aligned} p(\zeta) &= \int\limits_{B(0,2N)} s(z)\psi_1(\zeta)\chi_{CB(0,|z|)}(\zeta) K_\rho(z,\zeta) dm_2(z) \\ &= \int\limits_{B(0,|\zeta|)} s(z)\psi_1(\zeta) K_\rho(z,\zeta) dm_2(z). \end{aligned}$$

Note that $p(\zeta) = 0$ for $\zeta = 0$. Applying the reasoning in theorem 5 after the equality (11), we obtain that the function $p(\zeta)$ is continuous on the set $B(0, 2N)$.

Note that the equality $p(\zeta) = 0$ holds for $|\zeta| = 2N$. If we assume that $p(\zeta) = 0$ for $|\zeta| = 2N$, then the equality (16) can be rewritten as

$$J_{1,n} = \int p(\zeta)d\alpha_n(\zeta), \tag{17}$$

where $p(\zeta)$ is a continuous function compactly supported in $\mathbb{C}$.

From the condition of the lemma it follows that $\lim\limits_{n\to\infty} J_{1,n} = 0$. From this and the inequality (15) we have

$$\begin{aligned} \overline{\lim_{n\to\infty}} B_n &\le \overline{\lim_{n\to\infty}} J_{1,n} + \overline{\lim_{n\to\infty}} J_{2,n} = \overline{\lim_{n\to\infty}} J_{2,n} \\ &\le M(\rho, d) \overline{\lim_{n\to\infty}} \int\limits_N^\infty \frac{d|\alpha_n|(B(0,t))}{t^{\rho+1}}, \end{aligned} \tag{18}$$

where $M(\rho, d)$ is a constant depending only on $\rho$, $d$. Integrating by parts in the last integral, we obtain

$$\int\limits_N^\infty \frac{d|\alpha_n|(B(0,t))}{t^{\rho+1}} = \frac{|\alpha_n|(B(0,N))}{N^{\rho+1}} + \int\limits_N^\infty \frac{|\alpha_n|(B(0,t))}{t^{\rho+2}} dt.$$

The latter equality along with $|\alpha_n|(B(0,t)) \le Mt^\rho\gamma(t)$ and the inequality (18) imply that $B_n \to 0$ $(N \to \infty)$. The proof is complete.

**Theorem 7** *Let $\rho(r)$ be a proximate order with non-integer $\rho = \rho(\infty) > 0$, $p = [\rho]$. Let the measure $\mu$ be a measure of no higher than normal type with respect to the proximate order $\rho(r)$, which does not load the disk $B(0, 1)$. Let the sequence of measures $\mu_{t_n}$ $(t_n \to \infty)$ widely converge to the measure $\nu$. Then the sequence of functions*

$$\upsilon_n(z) = \int K_p(z, \zeta)d\mu_{t_n}(\zeta)$$

*converges to a function*

$$\upsilon(z) = \int K_p(z, \zeta)d\nu(\zeta)$$

*in the spaces $L_{1,loc}(\mathbb{C})$.*

*Proof.* We have

$$\upsilon_n(z) = \int\limits_{B(0,|z|)} K_p(z, \zeta)d\mu_{t_n}(\zeta) + \int\limits_{CB(0,|z|)} K_p(z, \zeta)d\mu_{t_n}(\zeta).$$

The result follows from theorems 5, 6.

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