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A Multiplicative Representation of the Resolvent Matrix of the Truncated Hausdorff Matrix Moment Problem via New Dyukarev-Stieltjes Parameters

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A new multiplicative decomposition of the resolvent matrix of the truncated Hausdorff matrix moment (THMM) problem in the case of an odd and even number of moments via new Dyukarev-Stieltjes matrix (DSM) parameters is attained. Additionally, we derive Blaschke-Potapov factors of auxiliary resolvent matrices; each factor is decomposed with the help of the DSM parameters.

Keywords: Orthogonal matrix polynomial; Dyukarev-Stieltjes parameter; Resolvent matrix; Continued fractions.

Абдон Чоке-Ріверо. **Мультиплікативне зображення резольвентної матриці усіченої матричної проблеми моментів Хаусдорфа в термінах нових параметрів Дюкарева-Стільтьєса.** Отримано мультиплікативний розклад резольвентної матриці усіченої матричної проблеми моментів Хаусдорфа у випадку непарного та парного числа моментів в термінах нових матричних параметрів Дюкарева-Стільтьєса. Крім того, ми перетворюємо множники Бляшке-Потапова допоміжних резольвентних матриць; кожний множник уявлено через параметри Дюкарева-Стільтьєса.

Ключові слова: ортогональні матричні многочлени; параметри Дюкарева-Стільтьєса; резольвентна матриця; неперервні дроби.

Абдон Чоке-Риверо. **Мультипликативное представление резольвентной матрицы усеченной матричной проблемы моментов Хаусдорфа в терминах новых параметров Дюкарева-Стилтьеса.** Получено мультипликативное разложение резольвентной матрицы усеченной матричной проблемы моментов Хаусдорфа в случае нечетного и четного числа моментов в терминах новых матричных параметров Дюкарева-Стилтьеса. Кроме того, мы преобразуем множители Бляшке-Потапова вспомогательных резольвентных матриц; каждый множитель выражается через параметры Дюкарева-Стилтьеса.

Ключевые слова: ортогональные матричные многочлены; параметры Дюкарева-Стилтьеса; резольвентная матрица; непрерывные дроби.
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1. Introduction

Throughout this paper, let q and p be positive integers. We will use $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$ and $\mathbb{N}$ to denote the set of all complex numbers, the set of all real numbers, the set of all nonnegative integers, and the set of all positive integers, respectively. The notation $\mathbb{C}^{q\times q}$ stands for the set of all complex $q\times q$ matrices. For the null matrix that belongs to $\mathbb{C}^{p\times q}$ we will write $0_{p\times q}$. We denote by 0_q and I_q the null and the identity matrices in $\mathbb{C}^{q\times q}$, respectively. In cases where the sizes of the null and the identity matrix are clear, we will omit the indices.

In the present work we introduce new matrix Stieltjes parameters, called Dyukarev-Stieltjes matrix (DSM) parameters of the truncated Hausdorff matrix moment (THMM) problem. With the help of the DSM parameters, we obtain a new multiplicative representation of the resolvent matrix (RM):

$$U^{(m)}(z)=\begin{pmatrix}\alpha^{(m)}(z) & \beta^{(m)}(z)\\ \gamma^{(m)}(z) & \delta^{(m)}(z)\end{pmatrix}$$

of the THMM problem in the case of an odd and even number of moments. The RM $U^{(m)}$ is a $2q\times 2q$ matrix polynomial, which we factorize as follows:

$$U^{(2n)}=\mathcal{D}_1\mathbb{l}_{-1}^{(2n)}\mathbb{m}_0^{(2n)}\dots\mathbb{m}_{n-1}^{(2n)}\mathbb{l}_{n-1}^{(2n)}\mathcal{B}_2^{(2n)}\mathcal{D}_2, \tag{1}$$

$$U^{(2n+1)}=\mathcal{D}_3\mathbb{l}_{-1}^{(2n+1)}\mathbb{m}_0^{(2n+1)}\dots\mathbb{l}_{n-1}^{(2n+1)}\mathbb{m}_n^{(2n+1)}\mathcal{B}_2^{(2n+1)}\mathbf{D}_1, \tag{2}$$

where $\mathcal{D}_k$ are anti-diagonal block matrices, $\mathbf{D}_1$ is a diagonal matrix, $\mathcal{B}_2^{(2n)}$, $\mathcal{B}_2^{(2n+1)}$, $\mathbb{l}_j^{(2n+1)}$, $\mathbb{m}_j^{(2n)}$ are constant anti-triangular block matrices and $\mathbb{m}_j^{(2n+1)}$, $\mathbb{l}_j^{(2n)}$ are affine on z and anti-triangular block matrices.

See Theorem 3 and Corollary 1.

The importance of the RM is explained by the fact that linear fractional transformation

$$s(z)=(\alpha^{(m)}(z)\mathbf{p}(z)+\beta^{(m)}(z)\mathbf{q}(z))(\gamma^{(m)}(z)\mathbf{p}(z)+\delta^{(m)}(z)\mathbf{q}(z))^{-1}$$

describes the set of all associated solutions in the nondegenerate case of the THMM problem. Here the column pair $(\mathbf{p},\mathbf{q})$ satisfies certain properties in every case; see Definitions [10, Definition 5.2] and [9, Definition 5.2].

Let us now summarize the notions appearing in the last two paragraphs.

Statement of the THMM problem. The THMM problem is stated as follows: given an interval $[a,b]$ on the real axis and a finite sequence of $q\times q$ matrices, $(s_j)_{j=0}^m$, describe the set $\mathcal{M}_{\geq}^q[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^m]$ of all nonnegative Hermitian $q\times q$ measures σ defined on the σ-algebra of all Borel subsets of the interval $[a,b]$ such that

$$s_j=\int\limits_{[a,b]} t^j d\sigma(t)$$

holds true for each integer jf with $0 \le j \le m$.

Solution set of the THMM problem. For describing the solution set of the THMM problem with the help of the finite sequence $(s_j)_{j=0}^{2n}$ (resp. $(s_j)_{j=0}^{2n+1}$), we construct the following Hankel matrices

$$\widetilde{H}_{0,j} := \{s_{l+k}\}_{l,k=0}^{j}, \quad \widetilde{H}_{1,j} := \{s_{l+k+1}\}_{l,k=0}^{j}, \quad \text{and} \quad \widetilde{H}_{2,j} := \{s_{l+k+2}\}_{l,k=0}^{j}. \tag{3}$$

Furthermore, denote

$$H_{1,j} := \widetilde{H}_{0,j},\ j \ge 0, \quad H_{2,j-1} := -ab\widetilde{H}_{0,j-1} + (a+b)\widetilde{H}_{1,j-1} - \widetilde{H}_{2,j-1},\ j \ge 1 \tag{4}$$

and

$$K_{1,j} := bH_{1,j} - \widetilde{H}_{1,j}, \quad K_{2,j} := -aH_{1,j} + \widetilde{H}_{1,j}, \quad j \ge 0. \tag{5}$$

In [9, Theorem 1.3] (resp. [10, Theorem 1.3]), it was demonstrated that there is a solution to the THMM problem, that is, the set $\mathcal{M}_{\ge}^{q}[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{2n}]$ (resp. $\mathcal{M}_{\ge}^{q}[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{2n+1}]$ is not empty if and only if the block matrices $H_{1,n}$ and $H_{2,n-1}$ (resp. $K_{1,n}$ and $K_{2,n}$) are both nonnegative Hermitian.

The problem of finding the set $\mathcal{M}_{\ge}^{q}[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{m}]$ for $m = 2n$ and $m = 2n+1$ is usually reduced to searching for the set of holomorphic functions

$$\begin{aligned} &\mathfrak{S}_{\ge}^{q}[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{m}] \\ &\quad := \left\{ s(z) = \int_{[a,b]} \frac{d\sigma(t)}{t-z},\ \sigma \in \mathcal{M}_{\ge}^{q}[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{m}] \right\}. \end{aligned}$$

Definition 1 *Let $[a,b]$ be a finite interval on real axis $\mathbb{R}$. The sequence $(s_k)_{k=0}^{2j}$ (resp. $(s_k)_{k=0}^{2j+1}$) is called a Hausdorff positive definite sequence if the block Hankel matrices $H_{1,j}$ and $H_{2,j-1}$ (resp. $K_{1,j}$ and $K_{2,j}$) are both positive definite matrices.*

In the sequel, we will consider only Hausdorff positive definite sequences. In this case the THMM problem is called a nondegenerate THMM problem.

Resolvent matrix of the THMM problem. In the present work we use the following form of RM of the nondegenerate THMM problem, introduced in [5, Formula (3.24)]:

$$U^{(2n)}(z,a,b) := \begin{pmatrix} \Theta_{2,n}^{*}(\bar z,a)\Theta_{2,n}^{*^{-1}}(a,a) & \frac{1}{b-a}\Theta_{1,n}^{*}(\bar z,b)\Gamma_{1,n}^{*^{-1}}(a,b) \\ (z-a)\Gamma_{2,n}^{*}(\bar z,a)\Theta_{2,n}^{*^{-1}}(a,a) & \frac{b-z}{b-a}\Gamma_{1,n}^{*}(\bar z,b)\Gamma_{1,n}^{*^{-1}}(a,b) \end{pmatrix} \tag{6}$$

and [5, Formula (3.27)]

$$\begin{aligned} &U^{(2n+1)}(z,a,b) \\ &:= \begin{pmatrix} Q_{2,n}^{*}(\bar z,a,b)Q_{2,n}^{*^{-1}}(a,b,a) & -Q_{1,n+1}^{*}(\bar z)P_{1,n+1}^{*^{-1}}(a) \\ -(z-a)(b-z)P_{2,n}^{*}(\bar z,a,b)Q_{2,n}^{*^{-1}}(a,b,a) & P_{1,n+1}^{*}(\bar z)P_{1,n+1}^{*^{-1}}(a) \end{pmatrix}. \end{aligned} \tag{7}$$

The $q\times q$ matrix polynomials $P_{k,j}$, $Q_{k,j}$, $\Gamma_{k,j}$ and $\Theta_{k,j}$ for $k=\{1,2\}$ are $q\times q$ are constructed via the given data: the sequence of moments $(s_j)_{j=0}^{2n}$ (resp. $(s_j)_{j=0}^{2n+1}$). See Definition 6 and 7.

It should be mentioned that the THMM problem in the nondegenerate case was first solved in [30].

Factorization strategy of the RM of the THMM problem. Our main purpose is to factorize the RM $U^{(2n)}$ and $U^{(2n+1)}$ as their simplest factors. To this end we pursue the following strategy consisting of three steps.

Step 1. We use the equality

$$U^{(2n)}(z)=\begin{pmatrix}\frac{1}{(z-a)(b-z)}I_q & 0_q\\ 0_q & I_q\end{pmatrix}\begin{pmatrix}I_q & s_0 z\\ 0_q & I_q\end{pmatrix} \cdot \widetilde{U}_2^{(2n-2)}(z)A_2^{(2n)}\begin{pmatrix}(b-a)(z-a)I_q & 0_q\\ 0_q & \frac{b-z}{b-a}I_q\end{pmatrix} \tag{8}$$

and

$$U^{(2n+1)}(z)=\begin{pmatrix}\frac{1}{b-z}I_q & 0_q\\ 0_q & I_q\end{pmatrix}\widetilde{U}_2^{(2n+1)}(z)A_2^{(2n+1)}\begin{pmatrix}(b-z)I_q & 0_q\\ 0_q & I_q\end{pmatrix}, \tag{9}$$

where $\widetilde{U}_2^{(m)}(z)$, $A_2^{(m)}(z)$ for $m=2n-2$ $(m=2n+1)$ are introduced in (27), (A.1), (28) and (31). Equalities (8) and (9) are the consequence of [9, Equality (6.26)] and [10, Equalities (6.26),(6.27)].

Step 2. The auxiliary matrix $\widetilde{U}_2^{(2n+1)}$ is written in the following form (as in Corollary 1):

$$\widetilde{U}_2^{(2n+1)}=d^{(1)}d^{(3)}\dots d^{(2n-1)}d^{(2n+1)}. \tag{10}$$

Instead of $\widetilde{U}_2^{(2n-2)}$, the auxiliary matrix $\widehat{U}_2^{(2n-2)}$ (as in (29)) is used. The factorization

$$\widehat{U}_2^{(2n-2)}=d^{(0)}d^{(2)}\dots d^{(2n-2)}d^{(2n)} \tag{11}$$

is employed to prove a new factorization of the RM $U^{(2n-2)}$. The $2q\times 2q$ matrices $d^{(2j+1)}$ and $d^{(2j)}$ are affine on z. See Definition 8.

Step 3. We factorize every matrix $d^{(2j+1)}$ (resp. $d^{(2j+2)}$) as in the Theorem 2:

$$d^{(2j+1)}(z)=\begin{pmatrix}I_q & \mathbf{r}_j\\ 0_q & I_q\end{pmatrix}\begin{pmatrix}I_q & 0_q\\ -(z-a)\mathbf{m}_j & I_q\end{pmatrix}\begin{pmatrix}I_q & -\mathbf{r}_j\\ 0_q & I_q\end{pmatrix}, \tag{12}$$

$$d^{(2j+2)}(z)=\begin{pmatrix}I_q & 0_q\\ -\mathbf{t}_j & I_q\end{pmatrix}\begin{pmatrix}I_q & (z-a)\mathbf{l}_j\\ 0_q & I_q\end{pmatrix}\begin{pmatrix}I_q & 0_q\\ \mathbf{t}_j & I_q\end{pmatrix} \tag{13}$$

for $0\le j\le n$ (resp. $0\le j\le n-1$) where the $q\times q$ matrices $\mathbf{r}_j$, $\mathbf{t}_j$, $\mathbf{m}_j$ and $\mathbf{l}_j$ are as in Definition 9.

Based on Steps 1 through 3, which involves algebraic identities and auxiliary results described in Sections 2–4, the multiplicative representations (106) and (107) are found.

In [8], a similar strategy was employed to attain another factorization of the RM $U^{(2n)}$ and $U^{(2n+1)}$. Namely, the following relations were used:

$$U^{(2n)} = \widetilde{U}_1^{(2n)} A^{(2n)}$$

and

$$U^{(2n+1)} = \begin{pmatrix} \frac{1}{z-a} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \widetilde{U}_1^{(2n+1)} A^{(2n+1)} \begin{pmatrix} (z-a) I_q & 0_q \\ 0_q & I_q \end{pmatrix}.$$

The auxiliary matrices $\widetilde{U}_1^{(2n+1)}$ and $\widetilde{U}_1^{(2n)}$ are defined by [8, Formula (1.14) and (1.32)]. The symbol $A^{(2n)}$ (resp. $A^{(2n+1)}$) denotes a $2q \times 2q$ matrix depending on a and b.

Observe that the auxiliary matrices $\widetilde{U}_1^{(2n)}$ and $\widetilde{U}_1^{(2n+1)}$ (resp. $\widetilde{U}_2^{(2n-2)}$ and $\widetilde{U}_2^{(2n+1)}$) are related to $H_{1,n}$ and $K_{2,n}$ (resp. $H_{2,n-1}$ and $K_{1,n}$), correspondingly.

The importance of the auxiliary matrices $\widetilde{U}_2^{(2n+1)}$ and $\widehat{U}_2^{(2n-2)}$ resides in the fact that they belong to the Potapov class of matrix functions [42], [10, Lemma 6.3], [9, Proposition 6.3]:

Definition 2 *Let* $J_q := \begin{pmatrix} 0_q & iI_q \\ -iI_q & 0_q \end{pmatrix}$. *Furthermore, let* $\Pi_+ := \{w \in \mathbb{C} : \operatorname{Im} w \in (\infty, 0)\}$. *A matrix-valued entire function* $W : \mathbb{C} \mapsto \mathbb{C}^{p \times p}$ *is said to belong to the Potapov class* $\mathfrak{P}_{J_q}(\Pi_+)$ *if*

$$J_q - W^*(z) J_q W(z) \geq 0$$

is satisfied for all $z \in \Pi_+$. *A matrix-valued function* W *that belongs to* $\mathfrak{P}_{J_q}(\Pi_+)$ *is called a J-inner function of* $\mathfrak{P}_{J_q}(\Pi_+)$ *if*

$$J_q - W^*(x) J_q W(x) = 0$$

holds for all $x \in \mathbb{R}$.

Matrix-valued functions belonging to the Potapov class can be factorized into elementary factors, as seen in Corollary 2.

The determinateness of the TSMM problem was obtained in [25] with the help of the Dyukarev-Stieltjes matrix parameters of the TSMM. The results obtained in [25] were generalized in [26], [27], [28], [29] and [33]. In these papers, the Yu.M. Dyukarev's factorization of the matrix valued functions in the Stieltjes class [24] and [23] were employed.

In [31], by using a decomposition of the RM of the TSMM problem, the following were demonstrated: necessary and sufficient conditions for the TSMM problem to have a unique solution and infinitely many solutions for the Hamburger moment problem with the same moments. Note that in [47] and [14] the operator approach was employed to solve the THMM problem.

In comparison to the DSM parameters $\mathbf{M}_k$ and $\mathbf{L}_k$ [8], the new DSM parameters $\mathbf{m}_j$ and $\mathbf{l}_j$ depend on both terminal points of the interval $[a, b]$. Other DSM parameters which also depend on a and b were introduced in [4]. In turn the aforementioned parameters are different from the ones studied in [8] (also in [3]), where the parameters depend only on a. In Remark 8 by setting $b \to +\infty$ and $a = 0$ in the DSM parameters $\mathbf{m}_j$ and $\mathbf{l}_j$, we obtain the Dyukarev-Stieltjes parameters of the TSMM problem [25].

Throughout the paper we decisively use the forms (6) and (7) of the RM of the THMM problem obtained in [5] where the elements of the RM are given with the help of four orthogonal polynomials and their second kind polynomials. Orthogonal matrix polynomials (OMP) were first considered by M.G. Krein in 1949 [39], [40]. Further investigations of OMP were made by J.S. Geronimo [36], I.V. Kovalishina [37], [38], H. Dym [22], B. Simon [44], Damanik/Pushnitski/-Simon [15] and the references therein. See also [17], [18], [19], [20], [21], [34], [16], [41], [45], [31], [12], [13], [11], [6] and [7].

2. Notations and preliminaries

In this section we introduce some matrix notation which appear throughout the work. In particular, we propose the auxiliary RM $\widehat{U}_2^{(2j)}$ which will be factorized by elementary matrices. See Corollary 2.

The orthogonal matrix polynomials $P_{k,j}$, $\Gamma_{k,j}$ on $[a, b]$ as well as their second kind polynomials $Q_{k,j}$, $\Theta_{k,j}$ are recalled. The mentioned matrix polynomials together with the connection between the auxiliary RM $\widetilde{U}_2^{(2j+1)}$, $\widehat{U}_2^{(2j)}$ and the RM $U^{(m)}$ play an important role in this work.

Auxiliary matrices

Let $R_j : \mathbb{C} \to \mathbb{C}^{(j+1)q\times(j+1)q}$ be given by

$$R_j(z) := (I_{(j+1)q} - zT_j)^{-1}, \quad j \geq 0, \tag{14}$$

with

$$T_0 := 0_q, \quad T_j := \begin{pmatrix} 0_{q\times jq} & 0_q \\ I_{jq} & 0_{jq\times q} \end{pmatrix}, \quad j \geq 1.$$

Let

$$v_0 := I_q, \quad v_j := \begin{pmatrix} I_q \\ 0_{jq\times q} \end{pmatrix} = \begin{pmatrix} v_{j-1} \\ 0_q \end{pmatrix}, \quad \forall j \geq 0. \tag{15}$$

Furthermore, let

$$y_{[j,k]} := \begin{pmatrix} s_j \\ s_{j+1} \\ \vdots \\ s_k \end{pmatrix}, \quad 0 \le j \le k \le 2n. \tag{16}$$

Let

$$\begin{aligned} \widetilde{u}_{1,0} &:= s_0, \quad \widetilde{u}_{2,0} := -s_0, \\ \widetilde{u}_{1,j} &:= y_{[0,j]} - b \begin{pmatrix} 0_q \\ y_{[0,j-1]} \end{pmatrix}, \quad \widetilde{u}_{2,j} := -y_{[0,j]} + a \begin{pmatrix} 0_q \\ y_{[0,j-1]} \end{pmatrix} \end{aligned} \tag{17}$$

for every $1 \le j \le n-1$. In addition, for $1 \le j \le n$ let

$$\widetilde{Y}_{1,j} := by_{[j,2j-1]} - y_{[j+1,2j]}, \quad \widetilde{Y}_{2,j} := -a\, y_{[j,2j-1]} + y_{[j+1,2j]}. \tag{18}$$

Let $\widehat{K}_{1,j}$ (resp. $\widehat{K}_{2,j}$) denote the Schur complement of the block $bs_{2j} - s_{2j+1}$ (resp. $-as_{2j} + s_{2j+1}$) of the matrix $K_{1,j}$ (resp. $K_{2,j}$). In addition, denote

$$\widehat{K}_{1,0} = bs_0 - s_1,\ \widehat{K}_{1,j} := bs_{2j} - s_{2j+1} - \widetilde{Y}^*_{1,j} K^{-1}_{1,j-1} \widetilde{Y}_{1,j},\ 1 \le j \le n, \tag{19}$$

$$\widehat{K}_{2,0} = -as_0 + s_1,\ \widehat{K}_{2,j} := -\,as_{2j} + s_{2j+1} - \widetilde{Y}^*_{2,j} K^{-1}_{2,j-1} \widetilde{Y}_{2,j},\ 1 \le j \le n. \tag{20}$$

The quantities (19) and (20) have been defined in [16] for $a = 0$ and $b = 1$.

Let

$$u_{1,0} := 0_q,\ u_{1,j} := \begin{pmatrix} 0_q \\ -y_{[0,j-1]} \end{pmatrix}, \quad 1 \le j \le n \tag{21}$$

and

$$u_{2,0} := -(a+b)s_0 + s_1, \quad u_{2,j} := \begin{pmatrix} u_{2,0} \\ -\widehat{y}_{[0,j-2]} \end{pmatrix}, 1 \le j \le 2n. \tag{22}$$

Moreover, let

$$\widehat{s}_j := -abs_j + (a+b)s_{j+1} - s_{j+2}, \quad 0 \le j \le 2n-2 \tag{23}$$

and

$$\widehat{y}_{[j,k]} := \begin{pmatrix} \widehat{s}_j \\ \widehat{s}_{j+1} \\ \vdots \\ \widehat{s}_k \end{pmatrix}, 0 \le j \le k \le 2n-2.$$

Note that by (16) and (23)

$$\widehat{y}_{[j,k]} = -aby_{[j,k]} + (a+b)y_{[j+1,k+1]} - y_{[j+2,k+2]}.$$

We also denote

$$Y_{1,j} := y_{[j,2j-1]},\ 1 \le j \le n, \quad Y_{2,j} := \widehat{y}_{[j,2j-1]},\ 1 \le j \le n-1. \tag{24}$$

Let $\widehat{H}_{1,j}$ (resp. $\widehat{H}_{2,j}$) denote the Schur complement of the block s_{2j} (resp. $\widehat{s}_{2j-2}$) of the matrix $H_{1,j}$ (resp. $H_{2,j}$): denote $\widehat{H}_{1,0} = s_0$, $\widehat{H}_{2,0} = \widehat{s}_0$ and

$$\widehat{H}_{1,j} := s_{2j} - Y_{1,j}^* H_{1,j-1}^{-1} Y_{1,j}, \ 1 \le j \le n, \tag{25}$$

$$\widehat{H}_{2,j} := \widehat{s}_{2j} - Y_{2,j}^* H_{2,j-1}^{-1} Y_{2,j}, \ 1 \le j \le n-1. \tag{26}$$

The quantities (25) and (26) have been defined in [16] for $a = 0$ and $b = 1$.

In the following Definition, we recall the auxiliary RM $\widetilde{U}_2^{(2j+1)}$ introduced in [10, Formula (6.2)]. An additive expansion of the aforementioned matrix is attained in Proposition 3. In Corollary 2, a multiplicative representation of the auxiliary RM $\widetilde{U}_2^{(2j+1)}$ is achieved.

Definition 3 *Let $K_{1,j}$ be as in (5), and assume that $K_{1,j}$ is a positive definite matrix. Furthermore, let $\widetilde{u}_{1,j}$, R_j and v_j be as in (17), (14) and (15). The $2q \times 2q$ matrix polynomial*

$$\widetilde{U}_2^{(2j+1)}(z,a,b) := \begin{pmatrix} \widetilde{\alpha}_2^{(2j+1)}(z) & \widetilde{\beta}_2^{(2j+1)}(z) \\ \widetilde{\gamma}_2^{(2j+1)}(z) & \widetilde{\delta}_2^{(2j+1)}(z) \end{pmatrix}, \ z \in \mathbb{C}, \quad 1 \le j \le n, \tag{27}$$

with

$$\widetilde{\alpha}_2^{(2j+1)}(z,a,b) := I_q - (z-a)\widetilde{u}_{1,j}^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) v_j,$$
$$\widetilde{\beta}_2^{(2j+1)}(z,a,b) := (z-a)\widetilde{u}_{1,j}^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) \widetilde{u}_{1,j},$$
$$\widetilde{\gamma}_2^{(2j+1)}(z,a,b) := -(z-a) v_j^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) v_j$$

and

$$\widetilde{\delta}_2^{(2j+1)}(z,a,b) := I_q + (z-a) v_j^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) \widetilde{u}_{1,j}$$

is called the second auxiliary matrix of the THMM problem in the case of an even number of moments.

In [10], Equality (9) was proved by using $B_{2,j} := (b-a)\widetilde{u}_{2,j}^* R_j^*(a) K_{2,j}^{-1} R_j(a) \widetilde{u}_{2,j}$ and

$$A_2^{(2j+1)} := \begin{pmatrix} I_q & B_{2,j} \\ 0_q & I_q \end{pmatrix}. \tag{28}$$

In the subsequent Definition, we introduce the auxiliary RM $\widehat{U}_2^{(2j)}$. In Proposition 2, an additive expansion of the indicated matrix is attained. A multiplicative representation of the auxiliary RM $\widehat{U}_2^{(2j)}$ is given by equality (105).

Definition 4 *Let $H_{2,j}$ be as in (4), and assume that $H_{2,j}$ is a positive definite matrix. Furthermore, let $u_{2,j}$, R_j and v_j be as in (22), (14) and (15). The $2q \times 2q$ matrix polynomial*

$$\widehat{U}_2^{(2j)}(z,a,b) := \begin{pmatrix} \widehat{\alpha}_2^{(2j)}(z,a,b) & \widehat{\beta}_2^{(2j)}(z,a,b) \\ \widehat{\gamma}_2^{(2j)}(z,a,b) & \widehat{\delta}_2^{(2j)}(z,a,b) \end{pmatrix}, \; z \in \mathbb{C}, \quad 0 \le j \le n-1, \quad (29)$$

with

$$\begin{aligned} \widehat{\alpha}_2^{(2j)}(z,a,b) :=& I_q - (z-a)(u_{2,j}^* + zs_0 v_j^*)R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)v_j, \\ \widehat{\beta}_2^{(2j)}(z,a,b) :=& (z-a)(s_0 + (u_{2,j}^* + zs_0 v_j^*)R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)(u_{2,j} + av_j s_0)), \\ \widehat{\gamma}_2^{(2j)}(z,a,b) :=& -(z-a)v_j^* R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)v_j \end{aligned}$$

and

$$\widehat{\delta}_2^{(2j)}(z,a,b) := I_q + (z-a)v_j^* R_j^*(\bar{z})H_{2,j}^{-1}R_j(a)(u_{2,j} + av_j s_0)$$

is called the second transformed auxiliary matrix of the THMM problem in the case of an odd number of moments. The adjective transformed in the sequel will be omitted.

Let

$$N_{2,j} := -(b-a)^{-1}v_j^* R_j^*(a)H_{1,j}^{-1}R_j(a)v_j \quad (30)$$

and

$$A_2^{(2j)} := \begin{pmatrix} I_q & -as_0 \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ N_{2,j} & I_q \end{pmatrix}. \quad (31)$$

Remark 1 *Let $(s_j)_{j=0}^{2j}$ be a Hausdorff positive definite sequence and let $U^{(2j)}$, $\widetilde{U}_2^{(2j)}$ and $\widehat{U}_2^{(2j)}$ be as in (6), (A.1) and (29). The following equalities are valid:*
a)

$$\widehat{U}_2^{(2j)}(z) = \begin{pmatrix} I_q & zs_0 \\ 0_q & I_q \end{pmatrix} \widetilde{U}_2^{(2j)}(z) \begin{pmatrix} I_q & -as_0 \\ 0_q & I_q \end{pmatrix} \quad (32)$$

and b)

$$\begin{aligned} U^{(2j)}(z) = & \begin{pmatrix} \frac{1}{(z-a)(b-z)} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \widehat{U}_2^{(2j-2)}(z) \begin{pmatrix} I_q & 0_q \\ N_{2,j} & I_q \end{pmatrix} \\ & \cdot \begin{pmatrix} (z-a)(b-z)I_q & 0_q \\ 0_q & \frac{b-z}{b-a} I_q \end{pmatrix}. \end{aligned} \quad (33)$$

P r o o f. Equalities (32) and (33) readily follow by direct calculations.

Orthogonal matrix polynomials on $[a, b]$

Let us reproduce some notions on OMP which were introduced in [12]. Let P be a complex $p \times q$ matrix polynomial. For all $n \in \mathbb{N}_0$, let

$$Z_n^{[P]} := [A_0, A_1, \dots, A_n],$$

where $(A_j)_{j=0}^{\infty}$ is the unique sequence of complex $p \times q$ matrices such that for all $z \in \mathbb{C}$ the polynomial P admits the representation $P(z) = \sum_{j=0}^{\infty} z^j A_j$. Furthermore, we denote by $\deg P := \sup\{j \in \mathbb{N}_0 : A_j \neq 0_{p\times q}\}$ the *degree of* P. Observe that in the case $P(z) = 0_{p\times q}$ for all $z \in \mathbb{C}$ we thus have $\deg P = -\infty$. If $k := \deg P \geq 0$, we refer to A_k as the *leading coefficient of* P. For all $k \in \mathbb{N}_0$ and all $\kappa \in \mathbb{N}_0$ with $k \leq \kappa$, let $\mathbb{Z}_{k,\kappa} := \{n \in \mathbb{N}_0, k \leq n \leq \kappa\}$.

Definition 5 *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{2\kappa}$ be a sequence of complex $q \times q$ matrices. A sequence $(P_k)_{k=0}^{\kappa}$ of complex $q \times q$ matrix polynomials is called a* monic left orthogonal system of matrix polynomials with respect to $(s_j)_{j=0}^{2\kappa}$ *if the following three conditions are fulfilled:*

(I) $\deg P_k = k$ *for all* $k \in \mathbb{Z}_{0,\kappa}$*;*

(II) P_k *has the leading coefficient* I_q *for all* $k \in \mathbb{Z}_{0,\kappa}$*;*

(III) $Z_n^{[P_j]} H_n (Z_n^{[P_k]})^* = 0_{q\times q}$ *for all* $j, k \in \mathbb{Z}_{0,\kappa}$ *with* $j \neq k$*, where* $n := \max\{j, k\}$*.*

Remark 2 *[12, Remark 3.6] Let $n \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{2n}$ be a Hausdorff positive definite sequence: the corresponding Hankel block matrix H_n is positive definite. Denote by $(P_k)_{k=0}^{n}$ the monic left orthogonal system of matrix polynomials with respect to $(s_j)_{j=0}^{2n}$. Let σ be a nonnegative Hermitian $q \times q$ measure on $\mathbb{R}$ satisfying $s_j = \int_{[a,b]} t^j d\sigma(t)$ for $0 \leq j \leq 2n$. Thus,*

$$\int_{[a,b]} P_j d\sigma P_k^* = \begin{cases} \widehat{H}_j, & \text{if } j = k, \\ 0_q, & \text{if } j \neq k \end{cases}$$

for all $0 \leq j, k \leq n$ where $\widehat{H}_j$ denotes the Schur complement of H_{j-1} in H_j; see (25).

In the following two Definitions, we recall matrix-valued polynomials $P_{k,j}$, $Q_{k,j}$, $\Gamma_{k,j}$ and $\Theta_{k,j}$. With their help the RM $U^{(2n)}$ and $U^{(2n+1)}$ (as in (6), (7)) as well as the solution set of the THMM were described (as in [5, Propositions 4.4 and 4.5]).

Definition 6 *Let $(s_k)_{k=0}^{2j}$ be a Hausdorff positive definite sequence. Furthermore, let $H_{k,j}$, $u_{k,j}$, $Y_{k,j}$, for $k = 1, 2$, R_j and v_j be as in (4), (21), (22), (24), (14) and*

(15), respectively. Let

$$\begin{aligned} P_{1,0}(z) &:= I_q, \quad P_{2,0}(z) := I_q, \quad Q_{1,0}(z) := 0_q, \quad Q_{2,0}(z,a,b) := -(u_{2,0} + z\, s_0), \\ P_{1,j}(z) &:= (-Y_{1,j}^* H_{1,j-1}^{-1}, I_q) R_j(z) v_j, \quad 1 \le j \le n, \qquad (34) \\ P_{2,j}(z,a,b) &:= (-Y_{2,j}^* H_{2,j-1}^{-1}, I_q) R_j(z) v_j, \quad 1 \le j \le n-1, \qquad (35) \\ Q_{1,j}(z) &:= -(-Y_{1,j}^* H_{1,j-1}^{-1}, I_q) R_{1,j}(z) u_{1,j}, \quad 1 \le j \le n \end{aligned}$$

and

$$Q_{2,j}(z,a,b) := -(-Y_{2,j}^* H_{2,j-1}^{-1}, I_q) R_j(z)(u_{2,j} + z v_j s_0), \quad 1 \le j \le n-1. \qquad (36)$$

Definition 7 *Let $(s_k)_{k=0}^{2j+1}$ be a Hausdorff positive definite sequence. Furthermore, let $K_{k,j}$, $\widetilde{u}_{k,j}$, $\widetilde{Y}_{k,j}$, for $k = 1, 2$, R_j and v_j be as in (5), (17), (18), (14) and (15), respectively. Let*

$$\Gamma_{1,0}(z) := I_q,\, \Gamma_{2,0}(z) := I_q, \quad \Theta_{1,0}(z) := s_0,\, \Theta_{2,0}(z) := -s_0$$

for all $z \in \mathbb{C}$. For $k \in \{1, 2\}$ and $1 \le j \le n$, define

$$\begin{aligned} \Gamma_{1,j}(z,b) &:= (-\widetilde{Y}_{1,j}^* K_{1,j-1}^{-1}, I_q) R_j(z) v_j, \qquad (37) \\ \Gamma_{2,j}(z,a) &:= (-\widetilde{Y}_{2,j}^* K_{2,j-1}^{-1}, I_q) R_j(z) v_j, \qquad (38) \\ \Theta_{1,j}(z,b) &:= (-\widetilde{Y}_{1,j}^* K_{1,j-1}^{-1}, I_q) R_j(z) \widetilde{u}_{1,j} \qquad (39) \end{aligned}$$

and

$$\Theta_{2,j}(z,a) := (-\widetilde{Y}_{2,j}^* K_{2,j-1}^{-1}, I_q) R_j(z) \widetilde{u}_{2,j}, \qquad (40)$$

for all $z \in \mathbb{C}$.

For $k = 1, 2$, we usually omit the dependence of the polynomials $P_{k,j}$, $Q_{k,j}$, $\Gamma_{k,j}$ and $\Theta_{k,j}$ on the parameters a and b.

In [2], (resp. [46]) it was proved that polynomials $P_{k,j}$ (resp. $\Gamma_{k,j}$) for $k = 1, 2$ are in fact OMP on $[a, b]$. In [12] explicit interrelations between $P_{k,j}$, $\Gamma_{k,j}$ and their second kind polynomials were studied.

For the sake of completeness in the following Remark, we reproduce explicit interrelations between the matrices $\widehat{H}_{k,j}$, $\widehat{K}_{k,j}$ and the polynomials $P_{1,j}$, $Q_{2,j}$, $\Gamma_{1,j}$, $\Theta_{2,j}$ considered in [5, Corollary 3.4] and [5, Corollary 3.10].

Remark 3 *Let $\widehat{H}_{k,j}$, $\widehat{K}_{k,j}$, for $k = 1, 2$, $P_{1,j}$, $Q_{2,j}$, $\Gamma_{1,j}$ and $\Theta_{2,j}$ be as in (25), (26), (19), (20) and Definitions 6 and 7, respectively. The following equalities then hold:*

$$\begin{aligned} \widehat{H}_{1,j} &= -\, P_{1,j}(a) \Theta_{2,j}^*(a), \quad \widehat{H}_{2,j} = -Q_{2,j}(a) \Gamma_{1,j+1}^*(a), \qquad (41) \\ \widehat{K}_{1,j} &= \Gamma_{1,j}(a) Q_{2,j}^*(a), \quad \widehat{K}_{2,j} = \Theta_{2,j}(a) P_{1,j+1}^*(a). \qquad (42) \end{aligned}$$

3. Algebraic identities

In this section we will single out essential identities concerning the block matrices introduced in Section . Let

$$L_{1,n} := (\delta_{j,k+1} I_q)_{\substack{j=0,\dots,n\\ k=0,\dots,n-1}} \quad \text{and} \quad L_{2,n} := (\delta_{j,k} I_q)_{\substack{j=0,\dots,n\\ k=0,\dots,n-1}}, \tag{43}$$

where $\delta_{j,k}$ is the Kronecker symbol with $\delta_{j,k} := 1$ if $j = k$ and $\delta_{j,k} := 0$ if $j \neq k$.

Furthermore, let

$$\Xi_{1,j}^{K} := \begin{pmatrix} -K_{1,j-1}^{-1} \widetilde{Y}_{1,j} \\ I_q \end{pmatrix} \tag{44}$$

and

$$\Xi_{2,j}^{H} := \begin{pmatrix} -H_{2,j-1}^{-1} Y_{2,j} \\ I_q \end{pmatrix}. \tag{45}$$

In the following two remarks important identities are attained which will we mainly use in the proof of Preposition 1. In turn Preposition 1 is employed in Theorem 1 which is the main result of Section .

Remark 4 *Let v_j, $L_{2,j}$, $\widetilde{u}_{1,j}$, R_j, T_j, $H_{1,j}$, $u_{1,j}$, $K_{1,j}$, $\Xi_{1,j}^{K}$ and $\Xi_{2,j}^{H}$ be defined as in (15), (43), (17), (14), (4), (21), (5), (44) and (45), respectively. Then the following identities are valid:*

$$v_{j-1} - L_{2,j}^{*} v_j = 0, \tag{46}$$

$$\widetilde{u}_{1,j-1} - L_{2,j}^{*} \widetilde{u}_{1,j} = 0, \tag{47}$$

$$L_{2,j} - R_j^{*^{-1}}(\bar{z}) L_{2,j} R_{j-1}^{*}(\bar{z}) = 0, \tag{48}$$

$$L_{2,j} L_{1,j}^{*} - T_j^{*} = 0, \tag{49}$$

$$H_{1,j} T_j^{*} - T_j H_{1,j} - u_{1,j} v_j^{*} + v_j u_{1,j} = 0, \tag{50}$$

$$T_j K_{1,j} \Xi_{1,j}^{K} = 0, \tag{51}$$

$$T_j H_{2,j} \Xi_{2,j}^{H} = 0. \tag{52}$$

P r o o f. Equalities (46), (47), (48), (49) are proved by direct calculations. Identity (50) was considered in [10, Proposition 2.1]. We prove equality (51). Let $\lambda_j := (0_q, 0_q, \dots, 0_q, I_q)$ be a $q \times jq$ matrix. Thus $T_j = \begin{pmatrix} T_{j-1} & 0_{jq} \times q \\ \lambda_j & 0_q \end{pmatrix}$. By using the last equality and equality

$$K_{1,j} = \begin{pmatrix} K_{1,j-1} & \widetilde{Y}_{1,j} \\ \widetilde{Y}_{1,j}^{*} & bs_{2j} - s_{2j+1} \end{pmatrix}, \tag{53}$$

we have

$$T_j K_{1,j} \Xi_{1,j}^{K} = \begin{pmatrix} T_{j-1} & 0_{jq\times q} \\ \lambda_j & 0_q \end{pmatrix} \begin{pmatrix} K_{1,j-1} & \widetilde{Y}_{1,j} \\ \widetilde{Y}_{1,j}^{*} & bs_{2j} - s_{2j+1} \end{pmatrix} \begin{pmatrix} -K_{1,j-1}^{-1} \widetilde{Y}_{1,j} \\ I_q \end{pmatrix}$$
$$= \begin{pmatrix} T_{j-1} & 0_{jq\times q} \\ \lambda_j & 0_q \end{pmatrix} \begin{pmatrix} 0_{jq\times q} \\ \widehat{K}_{1,j} \end{pmatrix} = \begin{pmatrix} 0_{jq\times q} \\ 0_q \end{pmatrix}.$$

Equality (52) can be proved in a similar manner to (51) with the aid of $H_{2,j} = \begin{pmatrix} H_{2,j-1} & Y_{2,j} \\ Y_{2,j}^* & \widehat{s}_{2j} \end{pmatrix}$, (44) in place of (53), (45), respectively.

Remark 5 *Let $u_{1,j}$, v_j, $H_{1,j}$, T_j, $L_{2,j}$, $L_{1,j}$, R_j, $\widetilde{H}_{2,j}$, $\widetilde{H}_{1,j}$ and $\widetilde{H}_{0,j}$ be defined as in (21), (15), (4), (43), (14) and (3), respectively. The following identities are valid:*

$$u_{1,j}^* + v_j^* H_{1,j} T_j^* = 0, \tag{54}$$
$$v_j^* H_{1,j} - v_{j+1}^* H_{1,j+1} L_{2,j+1} = 0, \tag{55}$$
$$T_{j+1}^* L_{1,j+1} + (T_{j+1}T_{j+1}^* - I)L_{2,j+1} - L_{2,j+1}L_{1,j}L_{1,j}^* = 0, \tag{56}$$
$$T_{j+1}^* L_{1,j+1} T_j^* - T_{j+1}^* L_{2,j+1} L_{1,j} L_{1,j}^* = 0, \tag{57}$$
$$T_{j+1}^* L_{2,j+1} - L_{2,j+1} L_{2,j} L_{1,j}^* = 0, \tag{58}$$
$$T_{j+1}^* L_{2,j+1} T_j^* - T_{j+1}^* L_{2,j+1} L_{2,j} L_{1,j}^* = 0, \tag{59}$$
$$(I - zT_{j+1}^*)L_{2,j+1}(T_j T_j^* - I) - (T_{j+1}T_{j+1}^* - I)L_{2,j+1} = 0, \tag{60}$$
$$(I - zT_{j+1}^*)L_{2,j+1}\left(I + (z-a)T_j^* R_j^*(\bar{z})\right) - (I - aT_{j+1}^*)L_{2,j+1} = 0, \tag{61}$$
$$v_j v_{j+2}^* H_{1,j+2} L_{1,j+2} - L_{1,j+1}^* H_{1,j+1} + L_{2,j+1}^* T_{j+1} \widetilde{H}_{2,j+1} = 0, \tag{62}$$
$$v_j v_{j+2}^* H_{1,j+2} L_{2,j+2} + L_{2,j+1}^* T_{j+1} \widetilde{H}_{1,j+1} - L_{2,j+1}^* H_{1,j+1} = 0, \tag{63}$$
$$T_j L_{1,j+1}^* H_{1,j+1} - L_{2,j+1}^* T_{1,j+1} \widetilde{H}_{1,j+1} = 0, \tag{64}$$
$$- L_{2,j+1}^* T_{j+1} \widetilde{H}_{0,j+1} + T_j L_{2,j+1}^* H_{1,j+1} = 0, \tag{65}$$
$$\begin{aligned} &- T_{j+1}^*(L_{1,j+1} - bL_{2,j+1}) - (T_{j+1}T_{j+1}^* - I)L_{2,j+1}R_j^*(a) \\ &+ (I - aT_{j+1}^*)L_{2,j+1}(L_{1,j} - bL_{2,j})L_{1,j}^* R_j^*(a) = 0. \end{aligned} \tag{66}$$

Proof. Identities (54)-(65) follow from a straightforward calculation. Equality (66) follows from (56)-(59).

Now we derive coupling identities between the block Hankel matrices $K_{1,j}$ and $K_{2,j}$ (resp. $H_{1,j}$ and $H_{2,j}$). These identities are crucially used in the proof of Theorem 1. Note that other coupling identities were attained in [10, Proposition 2.2], [9, Proposition 2.5] and [5, Proposition 6.2].

Proposition 1 *Let T_j, $L_{1,j}$, $L_{2,j}$, R_j, v_j, $H_{1,j}$, and $H_{2,j}$ be defined as in (43), (14), (15), (4) and (45), respectively. The following identities are valid:*

$$R_{j-1}^{-1}(a)K_{1,j-1}L_{1,j}^* - L_{2,j}^* K_{1,j} T_j^* + L_{2,j}^* T_j K_{1,j} - L_{2,j}^* T_j K_{2,j} {R_j^*}^{-1}(a) = 0, \tag{67}$$
$$\begin{aligned} &- R_j(a) v_j v_{j+2}^* H_{1,j+2}(L_{1,j+2} - bL_{2,j+2}) + (L_{1,j+1}^* - bL_{2,j+1}^*)H_{1,j+1} \\ &+ L_{2,j+1}^* R_{j+1}(a) T_{j+1} H_{2,j+1} = 0. \end{aligned} \tag{68}$$

Equality (67) follows by a straightforward calculation. Identity (68) follows from (62)-(65).

4. The Blaschke-Potapov factors

In this section we obtain a multiplicative representation (10), (11) of the second auxiliary matrices $\widetilde{U}_2^{(2n+1)}$ and $\widehat{U}_2^{(2n-2)}$ via the Blaschke-Potapov factors $d^{(2j+1)}$ and $d^{(2j)}$ defined in (73)-(75).

Since the matrices $H_{2,j}$ and $K_{1,j}$ are positive definite matrices for $0 \le j \le n-1$ and $0 \le j \le n$, respectively, their inverses can be written as

$$H_{2,j}^{-1} = \begin{pmatrix} H_{2,j-1}^{-1} & 0_{jq\times q} \\ 0_{q\times jq} & 0_q \end{pmatrix} + \begin{pmatrix} -H_{2,j-1}^{-1} Y_{2,j} \\ I_q \end{pmatrix} \widehat{H}_{2,j}^{-1}(-Y_{2,j}^* H_{2,j-1}^{-1}, I_q) \tag{69}$$

and

$$K_{1,j}^{-1} = \begin{pmatrix} K_{1,j-1}^{-1} & 0_{jq\times q} \\ 0_{q\times jq} & 0_{q\times q} \end{pmatrix} + \begin{pmatrix} -K_{2,j-1}^{-1} \widetilde{Y}_{1,j} \\ I_q \end{pmatrix} \widehat{K}_{1,j}^{-1}(-\widetilde{Y}_{1,j}^* K_{1,j-1}^{-1}, I_q). \tag{70}$$

Remark 6 *Due to Lemma 2 of the appendix, $\widehat{H}_{2,j}$ (resp. $\widehat{K}_{1,j}$) is a positive definite matrix if and only if $H_{2,j}$ (resp. $K_{1,j}$) is a positive definite matrix.*

In the following two propositions, we prove an additive property of the block elements of the auxiliary RM $\widehat{U}_2^{(2j)}(z)$ and $\widetilde{U}_2^{(2j+1)}(z)$. These properties give an indication in the form of the Blaschke-Potapov factors $d^{(2j)}$ and $d^{(2j+1)}$.

Proposition 2 *Let $H_{2,j}$ be as in (4), and assume that $H_{2,j}$ is a positive definite matrix. Furthermore, let the polynomials $P_{2,j}$ and $Q_{2,j}$ be as in Definition 6 and $\widehat{H}_{2,j}$ be defined as in (26). The block elements of the matrix $\widehat{U}_2^{(2j)}(z)$ defined by (29) can be written in the form*

$$\begin{aligned}
\widehat{\alpha}_2^{(2j)}(z) &= \widehat{\alpha}_2^{(2j-2)}(z) + (z-a)Q_{2,j}^*(\bar z)\widehat{H}_{2,j}^{-1}P_{2,j}(a),\\
\widehat{\beta}_2^{(2j)}(z) &= \widehat{\beta}_2^{(2j-2)}(z) + (z-a)Q_{2,j}^*(\bar z)\widehat{H}_{2,j}^{-1}Q_{2,j}(a),\\
\widehat{\gamma}_2^{(2j)}(z) &= \widehat{\gamma}_2^{(2j-2)}(z) - (z-a)P_{2,j}^*(\bar z)\widehat{H}_{2,j}^{-1}P_{2,j}(a)
\end{aligned}$$

and

$$\widehat{\delta}_2^{(2j)}(z) = \widehat{\delta}_2^{(2j-2)}(z) - (z-a)P_{2,j}^*(\bar z)\widehat{H}_{2,j}^{-1}Q_{2,j}(a).$$

Proof. Use (15), (69) and

$$R_j(z) = \left(\begin{array}{c|c} R_{j-1}(z) & 0_{jq\times q} \\ \hline (z^j I_q, z^{j-1} I_q, \dots, z I_q) & I_q \end{array}\right), \quad u_{2,j} = \begin{pmatrix} u_{2,j-1} \\ -\widehat{s}_{j-1} \end{pmatrix} \tag{71}$$

for $j \ge 2$.

Proposition 3 *Let $K_{1,j}$ be as in (5), and assume that $K_{1,j}$ is a positive definite matrix. Let the polynomials $\Theta_{1,j}$ and $\Gamma_{1,j}$ be as in Definition 7. The block elements of the matrix $\widetilde{U}_2^{(2j+1)}(z)$ defined by (27) can then be written in the form*

$$\widetilde{\alpha}_2^{(2j+1)}(z) = \widetilde{\alpha}_2^{(2j-1)}(z) - (z-a)\Theta^*_{1,j}(\bar{z})\widehat{K}^{-1}_{1,j}\Gamma_{1,j}(a),$$
$$\widetilde{\beta}_2^{(2j+1)}(z) = \widetilde{\beta}_2^{(2j-1)}(z) + (z-a)\Theta^*_{1,j}(\bar{z})\widehat{K}^{-1}_{1,j}\Theta_{1,j}(a),$$
$$\widetilde{\gamma}_2^{(2j+1)}(z) = \widetilde{\gamma}_2^{(2j-1)}(z) - (z-a)\Gamma^*_{1,j}(\bar{z})\widehat{K}^{-1}_{1,j}\Gamma_{1,j}(a)$$

and

$$\widetilde{\delta}_2^{(2j+1)}(z) = \widetilde{\delta}_2^{(2j-1)}(z) + (z-a)\Gamma^*_{1,j}(\bar{z})\widehat{K}^{-1}_{1,j}\Theta_{1,j}(a).$$

P r o o f. Use (15), (70), the first equality of (71) and equality

$$\widetilde{u}_{1,j} = \begin{pmatrix} \widetilde{u}_{1,j-1} \\ -bs_{j-1} + s_j \end{pmatrix} \tag{72}$$

for $j \geq 2$.

Definition 8 *Let $\widehat{H}_{2,j}$, $\widehat{K}_{1,j}$, $P_{2,j}$, $Q_{2,j}$, $\Theta_{1,j}$ and $\Gamma_{1,j}$ be as in (26), (19), and Definitions 6, 7, respectively. Define*

$$d^{(0)}(z) := \begin{pmatrix} I_q & (z-a)s_0 \\ 0_q & I_q \end{pmatrix}, \tag{73}$$

$$d^{(2j+2)}(z) := \begin{pmatrix} I_q + (z-a)Q^*_{2,j}(a)\widehat{H}^{-1}_{2,j}P_{2,j}(a) & (z-a)Q^*_{2,j}(a)\widehat{H}^{-1}_{2,j}Q_{2,j}(a) \\ -(z-a)P^*_{2,j}(a)\widehat{H}^{-1}_{2,j}P_{2,j}(a) & I_q - (z-a)P^*_{2,j}(a)\widehat{H}^{-1}_{2,j}Q_{2,j}(a) \end{pmatrix} \tag{74}$$

for $0 \leq j \leq n-1$, and

$$d^{(2j+1)}(z) := \begin{pmatrix} I_q - (z-a)\Theta^*_{1,j}(a)\widehat{K}^{-1}_{1,j}\Gamma_{1,j}(a) & (z-a)\Theta^*_{1,j}(a)\widehat{K}^{-1}_{1,j}\Theta_{1,j}(a) \\ -(z-a)\Gamma^*_{1,j}(a)\widehat{K}^{-1}_{1,j}\Gamma_{1,j}(a) & I_q + (z-a)\Gamma^*_{1,j}(a)\widehat{K}^{-1}_{1,j}\Theta_{1,j}(a) \end{pmatrix} \tag{75}$$

for $0 \leq j \leq n$.

The matrix function $d^{(2j)}$ (resp. $d^{(2j+1)}$) is called the Blaschke-Potapov factor of the auxiliary matrix $\widehat{U}_2^{(2k)}$ (resp. $\widetilde{U}_2^{(2k+1)}$).

Now we prove the main result of this section.

Theorem 1 *Let the matrix $\widehat{U}_2^{(2j)}$ (resp. $\widetilde{U}_2^{(2j+1)}$) be as in (29) (resp. (27)). Let $d^{(2j)}$, $d^{(2j+1)}$ be defined as in (73)-(75), then*

$$\widehat{U}_2^{(0)}(z) = d^{(0)}(z)d^{(2)}(z), \quad \widetilde{U}_2^{(1)}(z) = d^{(1)}(z), \tag{76}$$
$$\widehat{U}_2^{(2j)}(z) = \widehat{U}_2^{(2j-2)}(z)d^{(2j+2)}(z),\ z \in \mathbb{C}, \quad 1 \leq j \leq n-1 \tag{77}$$

and

$$\widetilde{U}_2^{(2j+1)}(z) = \widetilde{U}_2^{(2j-1)}(z)d^{(2j+1)}(z),\ z \in \mathbb{C}, \quad 1 \leq j \leq n. \tag{78}$$

P r o o f. Equality (76) readily follows from direct calculation. Now we demonstrate (77). Denote

$$G_j^{11}(a) := Q_{2,j}^*(a)\widehat{H}_{2,j}^{-1}P_{2,j}(a), \tag{79}$$
$$G_j^{12}(a) := Q_{2,j}^*(a)\widehat{H}_{2,j}^{-1}Q_{2,j}(a), \tag{80}$$
$$G_j^{21}(a) := P_{2,j}^*(a)\widehat{H}_{2,j}^{-1}P_{2,j}(a) \tag{81}$$

and

$$G_j^{22}(a) := P_{2,j}^*(a)\widehat{H}_{2,j}^{-1}Q_{2,j}(a) \tag{82}$$

for $1 \le j \le n-1$.

Now we prove equality (77). By using (73), (74), (79), (80), (81) and (82), Eq. (77) can be written in the equivalent form

$$\left(\begin{array}{cc} \widehat{\alpha}_2^{(2j)}(z) & \widehat{\beta}_2^{(2j)}(z) \\ \widehat{\gamma}_2^{(2j)}(z) & \widehat{\delta}_2^{(2j)}(z) \end{array}\right) - \left(\begin{array}{cc} \widehat{\alpha}_2^{(2j-2)}(z) & \widehat{\beta}_2^{(2j-2)}(z) \\ \widehat{\gamma}_2^{(2j-2)}(z) & \widehat{\delta}_2^{(2j-2)}(z) \end{array}\right)$$
$$\cdot \left(\begin{array}{cc} I_q + (z-a)G_j^{11}(a) & (z-a)G_j^{12}(a) \\ -(z-a)G_j^{21}(a) & I_q - (z-a)G_j^{22}(a) \end{array}\right) = 0. \tag{83}$$

The left-hand side of (83) is equivalent to the following four equalities:

$$\begin{aligned} \Upsilon_{11,j} :=& \widehat{\alpha}_2^{(2j)}(z) - \widehat{\alpha}_2^{(2j-2)}(z) \\ &+ (z-a)\left(-\widehat{\alpha}_2^{(2j-2)}(z)G_j^{11}(a) + \beta_2^{(2j-2)}(z)G_j^{21}(a)\right), \\ \Upsilon_{12,j} :=& \widehat{\beta}_2^{(2j)}(z) - \widehat{\beta}_2^{(2j-2)}(z) \\ &- (z-a)\left(\widehat{\alpha}_2^{(2j-2)}(z)G_j^{12}(a) - \beta_2^{(2j-2)}(z)G_j^{22}(a)\right), \\ \Upsilon_{21,j} :=& \widehat{\gamma}_2^{(2j+1)}(z) - \widehat{\gamma}_2^{(2j-2)}(z) \\ &+ (z-a)\left(-\widehat{\gamma}_2^{(2j-2)}(z)G_j^{11}(a) + \delta_2^{(2j-2)}(z)G_j^{21}(a)\right) \end{aligned}$$

and

$$\begin{aligned} \Upsilon_{22,j} :=& \widehat{\delta}_2^{(2j+1)}(z) - \widehat{\delta}_2^{(2j-2)}(z) \\ &- (z-a)\left(\widehat{\gamma}_2^{(2j-2)}(z)G_j^{12}(a) - \delta_2^{(2j-2)}(z)G_j^{22}(a)\right). \end{aligned}$$

By taking into account (79) and (81), we have

$$\begin{aligned} \Upsilon_{11,j} =& (z-a)\widetilde{\Upsilon}_{1,j}\widehat{H}_{2,j}^{-1}P_{2,j}(a), \quad \Upsilon_{12,j} = (z-a)\widetilde{\Upsilon}_{1,j}\widehat{H}_{2,j}^{-1}Q_{2,j}(a), \\ \Upsilon_{21,j} =& (z-a)\widetilde{\Upsilon}_{2,j}\widehat{H}_{2,j}^{-1}P_{2,j}(a), \quad \Upsilon_{22,j} = (z-a)\widetilde{\Upsilon}_{2,j}\widehat{H}_{2,j}^{-1}Q_{2,j}(a) \end{aligned}$$

where

$$\begin{aligned}\widetilde{\Upsilon}_{1,j+1} :=& Q^*_{2,j+1}(\bar z) - Q^*_{2,j+1}(a) + (z-a)(u^*_{2,j} + zs_0v^*_j)R^*_j(\bar z)H^{-1}_{2,j}R_j(a)v_j\\ &\cdot Q^*_{2,j+1}(a) + (z-a)(s_0 + (u^*_{2,j} + zs_0v^*_j)R^*_j(\bar z)H^{-1}_{2,j}R_j(a)\\ &\cdot (u_{2,j} + av_js_0))P^*_{2,j+1}(a),\\ \widetilde{\Upsilon}_{2,j+1} :=& -P^*_{2,j+1}(\bar z) + P^*_{2,j+1}(a) + (z-a)v^*_jR^*_j(\bar z)H^{-1}_{2,j}R_j(a)v_jQ^*_{2,j+1}(a)\\ &+ (z-a)v^*_jR^*_j(\bar z)H^{-1}_{2,j}R_j(a)(u_{2,j} + av_js_0)P^*_{2,j+1}(a).\end{aligned}$$

Now we verify that

$$\widetilde{\Upsilon}_{\ell,j} = 0, \quad \ell, k \in \{1,2\}, \quad 1 \le j \le n-1. \tag{84}$$

By using (36), (35) and (45), we have

$$\begin{aligned}\widetilde{\Upsilon}_{1,j+1} =& \big(-(u_{2,j+1} + zs_0v^*_{j+1})R^*_{j+1}(\bar z) + (u_{2,j+1} + as_0v^*_{j+1})R^*_{j+1}(a) +\\ &-(z-a)(u_{2,j} + zs_0v^*_j)R^*_j(\bar z)H^{-1}_{2,j}R_j(a)(u^*_{2,j+1} + as_0v^*_{j+1})R^*_{j+1}(a)\\ &+(z-a)\Big(s_0 + (u^*_{2,j} + zs_0v^*_j)R^*_j(\bar z)H^{-1}_{2,j}R_j(a)(u_{2,j} + av_js_0)\Big)\\ &\cdot v^*_{j+1}R^*_{j+1}(a)\big)\,\Xi^H_{2,j}\\ =&(z-a)v^*_{j+2}H_{1,j+2}R^*_{j+2}(\bar z)\big(-T^*_{j+2}(L_{1,j+2} - bL_{2,j+2})\\ &-\big((I - zT^*_{j+2})L_{2,j+2}(L_{1,j+2} - bL_{2,j+1}) + (z-a)(I - zT^*_{j+1})L_{2,j+2}\\ &\cdot R^*_{j+1}(\bar z)T^*_{j+1}(L_{1,j+1} - bL_{2,j+1}))H^{-1}_{2,j}R_j(a)v_jv^*_{j+2}H_{1,j+2}\\ &\cdot(L_{1,j+1} - bL_{2,j+1}) - (I - zT^*_{j+1})L_{2,j+2}(T_{j+1}T^*_{j+1} - I)R^*_{j+1}(a)\\ &+\big((I - zT^*_{j+2})L_{2,j+2}(L_{1,j+1} - bL_{2,j+1}) + (z-a)(I - zT^*_{j+2})L_{2,j+2}\\ &\cdot R^*_{j+1}(\bar z)T^*_{j+1}(L_{1,j+1} - bL_{2,j+1})\big)\\ &\cdot\Big(L_{1,j+1}R^*_{j+1}(a) + H^{-1}_{2,j}(L^*_{1,j+1} - bL^*_{2,j+1})H_{1,j+1}\Big)\Big)\,\Xi^H_{2,j}\\ =&(z-a)v^*_{j+2}H_{1,j+2}R^*_{j+2}(\bar z)\big(-T^*_{j+2}(L_{1,j+1} - bL_{2,j+1}) - (T_{j+2}T^*_{j+2} - I)\\ &\cdot L_{2,j+2}R^*_{j+1}(a) + (I - aT^*_{j+1})L_{2,j+2}(L_{1,j+1} - bL_{2,j+1})L^*_{1,j+1}R^*_{j+1}(a)\\ &-(I - aT^*_{j+2})L_{2,j+2}(L_{1,j+1} - bL_{2,j+1})H^{-1}_{2,j}R_j(a)v_jv^*_{j+2}H_{1,j+2}\\ &\cdot(L_{1,j+1} - bL_{2,j+1}) + (I - aT^*_{j+2})L_{2,j+2}(L_{1,j+1} - bL_{2,j+1})H^{-1}_{2,j}\\ &\cdot(L^*_{1,j+1} - bL^*_{2,j+1})H_{1,j+1}\big)\,\Xi^H_{2,j}\\ =&-(z-a)v^*_{j+2}H_{1,j+2}R^*_{j+2}(\bar z)(I - aT^*_{j+2})L_{2,j+2}(L_{1,j+1} - bL_{2,j+1})H^{-1}_{2,j}\\ &\cdot L^*_{2,j+1}T_{j+1}H_{2,j+1}\Xi^H_{2,j}\\ =&0.\end{aligned}$$

In the second equality we used (54) and (55), in the third equality we employed (60) and (61). The penultimate equality follows from (66) and (68). The last equality follows from identity (52). Equality (84) for $\ell = 2$ is proved by using (35), (36), (46), (49), (68) and (52).

To prove (78), we used the following equalities:

$$\Theta^*_{1,j}(\bar z)-\Theta^*_{1,j}(a)+(z-a)\widetilde u^*_{1,j-1}R^*_{j-1}(\bar z)K^{-1}_{1,j-1}R_{j-1}(a)v_{j-1}\Theta^*_{1,j}(a)$$
$$-(z-a)\widetilde u^*_{1,j-1}R^*_{j-1}(\bar z)K^{-1}_{1,j-1}R_{j-1}(a)\widetilde u_{1,j-1}\Gamma^*_{1,j}(a)=0 \qquad (85)$$

and

$$\Gamma^*_{1,j}(\bar z)-\Gamma^*_{1,j}(a)+(z-a)\widetilde v^*_{j-1}R^*_{j-1}(\bar z)K^{-1}_{1,j-1}R_{j-1}(a)v_{j-1}\Theta^*_{1,j}(a)$$
$$-(z-a)\widetilde v^*_{j-1}R^*_{j-1}(\bar z)K^{-1}_{1,j-1}R_{j-1}(a)\widetilde u_{1,j-1}\Gamma^*_{1,j}(a)=0. \qquad (86)$$

In turn (85) and (86) are demonstrated by using (37), (39), (47), (44), (49), (46), (50), (67) and (51). The Theorem is proved.

Corollary 1 *Let the auxiliary matrices* $\widetilde U_2^{(2n+1)}$ *and* $\widehat U_2^{(2n-2)}$ *defined as in (27) and (29). Furthermore, let* $d^{(2j)}$, $d^{(2j+1)}$ *be as in (73)-(75). Then equalities (10) and (11) hold.*

The proof follows immediately from Theorem 1.

5. Representation of the RM via DSM parameters

In this section we introduce new DSM parameters; see Definition 9. In contrast to the DSM parameters introduced in [3] (see Definition 10) which depend on the left terminal point of the interval $[a,b]$, the new DSM parameters depend both on a and b.

With the help of the new DSM parameters and the OMP on $[a,b]$ we obtain a multiplicative representation of the RM $U^{(2n)}$, $U^{(2n+1)}$ of the THMM problem. This representation is a generalization of a similar one attained by Yu. Dyukarev in [25, Theorem 7].

Definition 9 *Let a and b be real numbers such that $a<b$. Let $H_{2,j}$, $K_{1,j}$, R_j, v_j, $u_{2,j}$ be defined by (4), (5), (14), (15), (17) and (22), respectively. Furthermore, let s_0, $H_{2,j}$, $K_{1,j}$ be positive definite matrices. For $1\le j\le n-1$, denote by*

$$\mathbf r_0:=s_0,\quad \mathbf r_j(a,b):=s_0+(u^*_{2,j}+as_0v^*_j)R^*_j(a)H^{-1}_{2,j}R_j(a)(u_{2,j}+av_js_0), \qquad (87)$$

$$\mathbf t_0(b):=v^*_0R^*_0(a)K^{-1}_{1,0}R_0(a)v_0,\quad \mathbf t_j(a,b):=v^*_jR^*_j(a)K^{-1}_{1,j}R_j(a)v_j, \qquad (88)$$

$$\mathbf l_{-1}:=s_0,\quad \mathbf l_0(a,b):=(u^*_{2,0}+as_0v^*_0)H^{-1}_{2,0}(u_{2,0}+av_0s_0), \qquad (89)$$

$$\mathbf l_j(a,b):=(u^*_{2,j}+as_0v^*_j)R^*_j(a)H^{-1}_{2,j}R_j(a)(u_{2,j}+av_js_0)$$
$$-(u^*_{2,j-1}+as_0v^*_{j-1})R^*_{j-1}(a)H^{-1}_{2,j-1}R_{j-1}(a)(u_{2,j-1}+av_{j-1}s_0) \qquad (90)$$

and

$$\mathbf m_0(b):=\mathbf t_0(b),$$
$$\mathbf m_j(a,b):=v^*_jR^*_j(a)K^{-1}_{1,j}R_j(a)v_j-v^*_{j-1}R^*_{j-1}(a)K^{-1}_{1,j-1}R_{j-1}(a)v_{j-1} \qquad (91)$$

for $1 \leq j \leq n$. The matrices $\mathbf{l}_j(a,b)$ and $\mathbf{m}_j(a,b)$ are called the second type Dyukarev-Stieltjes matrix parameters of the THMM problem.

Below we shall usually omit the dependence on a and b of the matrices (87)-(91).

Observe that from (69), (70), (71), (72), (36), (37), (39) and (35) the following identities are valid:

$$\mathbf{l}_j = Q^*_{2,j}(a)\widehat{H}^{-1}_{2,j}Q_{2,j}(a), \quad \mathbf{m}_j = \Gamma^*_{1,j}(a)\widehat{K}^{-1}_{1,j}\Gamma_{1,j}(a), \tag{92}$$

$$\mathbf{r}_j = \Gamma^{-1}_{1,j}(a)\Theta_{1,j}(a), \qquad \mathbf{t}_j = Q^{-1}_{2,j}(a)P_{2,j}(a). \tag{93}$$

Remark 7 *Let $\mathbf{r}_j$, $\mathbf{t}_j$, $\mathbf{l}_j$ and $\mathbf{m}_j$ be as in (87)-(91). Thus, the following equalities hold:*

$$\mathbf{l}_j = \mathbf{r}_{j+1} - \mathbf{r}_j, \quad j \geq 0, \tag{94}$$

$$\mathbf{m}_j = \mathbf{t}_j - \mathbf{t}_{j-1}, \quad j \geq 1. \tag{95}$$

Moreover, the matrices $\mathbf{l}_j$ and $\mathbf{m}_j$ are positive definite matrices.

Proof. Equalities (94)-(95) follow by direct calculation from (87)-(91).
By the second equality of (41) and the fact that $\widehat{K}_{1,j}$ and $\widehat{H}_{2,j}$ are positive definite matrices, we obtain that $\mathbf{l}_j$ and $\mathbf{m}_j$ are too.

Let us recall DSM parameters $\mathbf{M}_j$ and $\mathbf{L}_j$ first introduced in [3].

Definition 10 *Let a be a real number. Let $H_{1,j}$, $K_{2,j}$, R_j, v_j, $\widetilde{u}_{2,j}$ be defined by (4), (5), (14), (15) and (17), respectively. Furthermore, let $H_{1,j}$, $K_{1,j}$ be positive definite matrices. For $1 \leq j \leq n$, denote by*

$$\mathbf{M}_0(a) := s_0^{-1}, \quad \mathbf{L}_0(a) := \widetilde{u}^*_{2,0}K^{-1}_{2,0}\widetilde{u}_{2,0}, \tag{96}$$

$$\mathbf{M}_j(a) := v^*_j R^*_j(a)H^{-1}_{1,j}R_j(a)v_j - v^*_{j-1}R^*_{j-1}(a)H^{-1}_{1,j-1}R_{j-1}(a)v_{j-1}, \tag{97}$$

$$\mathbf{L}_j(a) := \widetilde{u}^*_{2,j}R^*_j(a)K^{-1}_{2,j}R_j(a)\widetilde{u}_{2,j} - \widetilde{u}^*_{2,j-1}R^*_{j-1}(a)K^{-1}_{2,j-1}R_{j-1}(a)\widetilde{u}_{2,j-1}. \tag{98}$$

The matrices $\mathbf{M}_j(a)$ and $\mathbf{L}_j(a)$ are called Dyukarev-Stieltjes matrix parameters of the THMM problem.

In the sequel we usually omit the dependence of the DSM parameters $\mathbf{M}_j$ and $\mathbf{L}_j$ on the parameter a.

It should be mentioned that the notion *Dyukarev-Stieltjes parameters* was first introduced by B. Fritzsche, B. Kirstein and C. Mädler in [35] for the TSMM problem, that is, for $\mathbf{M}_j(0)$ and $\mathbf{L}_j(0)$.

In [25, Theorem 7] Dyukarev introduced the Stieltjes parameters for the Stieltjes matrix moment problem which in our notations are given by $\mathbf{M}_0(0)$, $\mathbf{L}_0(0)$, $\mathbf{M}_j(0)$ and $\mathbf{L}_j(0)$. The following Remark gives the interrelation between the aforementioned Stieltjes parameters [25, Theorem 7] and the DSM parameters studied in the present work.

Remark 8 *Let $\mathbf{M}_j$ and $\mathbf{L}_j$ be the DSM parameters as in (96)-(97). Furthermore, let the DSM parameters be as in (89), (90) and (91). Thus, the following relations are valid:*

$$\mathbf{M}_j(0) = \lim_{b\to+\infty} b\,\mathbf{m}_j(0,b), \qquad \mathbf{L}_j(0) = \lim_{b\to+\infty} b^{-1}\mathbf{l}_j(0,b). \tag{99}$$

This Remark can be verified by direct calculations.

We continue considering Hausdorff positive definite sequences $(s_j)_{j=0}^{2n}$ and $(s_j)_{j=0}^{2n+1}$. The following Theorem shows an explicit representation between the Blaschke-Potapov factors $d^{(2j)}$, $d^{(2j+1)}$ and the matrices $\mathbf{r}_j$, $\mathbf{t}_j$, $\mathbf{l}_j$ and $\mathbf{m}_j$.

Theorem 2 *Let $d^{(2j)}$, $d^{(2j+1)}$ be as in (73)-(74) and $\mathbf{r}_j$, $\mathbf{t}_j$, $\mathbf{l}_j$, $\mathbf{m}_j$ be defined by (87)-(91), respectively. The identities (12) and (13) then hold for $0 \le j \le n$.*

Proof. We prove (12). For $j = 0$ the proof can be checked by a direct calculation. Let $1 \le j \le n$. Denote

$$d^{(2j+1)} := \begin{pmatrix} d_j^{11} & d_j^{12} \\ d_j^{21} & d_j^{22} \end{pmatrix}.$$

The relation (12) is equivalent to the following four equalities:

$$d_j^{11} - I_q + (z-a)\mathbf{r}_j\mathbf{m}_j = 0, \tag{100}$$

$$d_j^{12} - (z-a)\mathbf{r}_j\mathbf{m}_j\mathbf{r}_j = 0, \tag{101}$$

$$d_j^{21} + (z-a)\mathbf{m}_j = 0, \tag{102}$$

$$d_j^{22} - I_q - (z-a)\mathbf{m}_j\mathbf{r}_j = 0. \tag{103}$$

Let us now prove (100). By the $(1,1)$ element of $d^{(2j+1)}$, (92) and (93), we have

$$\begin{aligned} &d_j^{11} - I_q + (z-a)\mathbf{r}_j\mathbf{m}_j \\ &= -(z-a)\Theta_{1,j}^*(a)\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a) + (z-a)\Theta_{1,j}^*(a){\Gamma_{1,j}^*}^{-1}(a)\Gamma_{1,j}^*(a)\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a) \\ &= 0. \end{aligned}$$

The equalities (101) and (103) are proved in a similar way. Observe that (102) is verified by definition. To prove (13) one uses (74), the first equality of (92) and the second equality of (93). Thus Theorem 2 is proved. Let $n \in \mathbb{N}_0$, and let $A_0, \ldots, A_n$ be complex $q \times q$ matrices. Let

$$\overrightarrow{\prod_{j=0}^{n}} A_j = A_0A_1\cdots A_{n-1}A_n \quad \text{and} \quad \overleftarrow{\prod_{j=0}^{n}} A_j = A_nA_{n-1}\cdots A_1A_0$$

then denote the right and left product of the matrices $A_0, A_1, \ldots, A_n$, respectively.

The following Corollary readily yields by employing (94), (95), Theorem 2 and Corollary 1.

Corollary 2 *Let* $\widehat{U}_2^{(2n-2)}$ *and* $\widetilde{U}_2^{(2n+1)}$ *be as in (29) and (27), respectively. Furthermore, let* $\mathbf{m}_k$, $\mathbf{l}_k$, $\mathbf{t}_k$ *and* $\mathbf{r}_k$ *be as in Definition 9. Thus, the equalities*

$$\widehat{U}_2^{(2n-2)}$$
$$= \overrightarrow{\prod_{k=0}^{n-1}} \left[\begin{pmatrix} I_q & (z-a)\mathbf{l}_{k-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -\mathbf{m}_k & I_q \end{pmatrix} \right] \begin{pmatrix} I_q & (z-a)\mathbf{l}_{n-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ \mathbf{t}_{n-1} & I_q \end{pmatrix} \tag{104}$$

and

$$\widetilde{U}_2^{(2n+1)} = \overrightarrow{\prod_{k=0}^{n}} \left[\begin{pmatrix} I_q & \mathbf{l}_{k-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -(z-a)\mathbf{m}_k & I_q \end{pmatrix} \right] \begin{pmatrix} I_q & -\mathbf{r}_n \\ 0_q & I_q \end{pmatrix} \tag{105}$$

are valid.

Now we derive a new representation of the RM of the THMM problem via DSM parameters in both cases for odd and even number of moments. We also reproduce an analogue representation given in [8, Corollary 3].

Theorem 3 *Let* $P_{k,j}$, $Q_{k,j}$, $\Gamma_{k,j}$ *and* $\Theta_{k,j}$ *be as in Definitions 6 and 7. Let the RM* $U^{(2n)}$, $U^{(2n+1)}$ *of the THMM problem be as in (6), (7), respectively.*
a) Let $\mathbf{l}_k$, $\mathbf{m}_k$, *be as in Definition (89)-(91). Thus, the following representations of the RM in the case of odd and even number of moments hold*

$$U^{(2n)}(z,a,b)$$
$$= \begin{pmatrix} \frac{1}{(b-z)(z-a)} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \overrightarrow{\prod_{k=0}^{n-1}} \left[\begin{pmatrix} I_q & (z-a)\mathbf{l}_{k-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -\mathbf{m}_k & I_q \end{pmatrix} \right]$$
$$\cdot \begin{pmatrix} I_q & (z-a)\mathbf{l}_{n-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ Q_{2,n-1}^{-1}(a)P_{2,n-1}(a) + \frac{1}{b-a}\Theta_{2,n}^{-1}(a)\Gamma_{2,n}(a) & I_q \end{pmatrix}$$
$$\cdot \begin{pmatrix} (b-a)(z-a)I_q & 0_q \\ 0_q & \frac{b-z}{b-a} I_q \end{pmatrix}, \tag{106}$$

$$U^{(2n+1)}(z,a,b) = \begin{pmatrix} \frac{1}{b-z} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \overrightarrow{\prod_{k=0}^{n}} \left[\begin{pmatrix} I_q & \mathbf{l}_{k-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -(z-a)\mathbf{m}_k & I_q \end{pmatrix} \right]$$
$$\cdot \begin{pmatrix} I_q & -\Gamma_{1,n}^{-1}(a)\Theta_{1,n}(a) - (b-a)P_{1,n+1}^{-1}(a)Q_{1,n+1}(a) \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} (b-z)I_q & 0_q \\ 0_q & I_q \end{pmatrix}. \tag{107}$$

b) Moreover, let $\mathbf{M}_k$, $\mathbf{L}_k$ *be as in (96)-(98). Thus the following representations* *hold:*

$$U^{(2n)}(z,a,b) = \overrightarrow{\prod_{k=0}^{n-1}} \left[\begin{pmatrix} I_q & 0_q \\ -(z-a)\mathbf{M}_k & I_q \end{pmatrix} \begin{pmatrix} I_q & \mathbf{L}_k \\ 0_q & I_q \end{pmatrix} \right] \begin{pmatrix} I_q & 0_q \\ -(z-a)\mathbf{M}_n & I_q \end{pmatrix}$$
$$\cdot \begin{pmatrix} I_q & Q^*_{1,n}(a)P^{*^{-1}}_{1,n}(a) + \frac{1}{b-a}\Theta^*_{1,n}(a)\Gamma^{*^{-1}}_{1,n}(a) \\ 0_q & I_q \end{pmatrix} \tag{108}$$

and

$$U^{(2n+1)}(z,a,b) = \begin{pmatrix} \frac{1}{z-a}I_q & 0_q \\ 0_q & I_q \end{pmatrix} \overrightarrow{\prod_{k=0}^{n}} \left[\begin{pmatrix} I_q & 0_q \\ -\mathbf{M}_k & I_q \end{pmatrix} \begin{pmatrix} I_q & (z-a)\mathbf{L}_k \\ 0_q & I_q \end{pmatrix} \right]$$
$$\cdot \begin{pmatrix} I_q & 0_q \\ -\Gamma^*_{2,n}(a)\Theta^{*^{-1}}_{2,n}(a) - (b-a)P^*_{2,n}(a)Q^{*^{-1}}_{2,n}(a) & I_q \end{pmatrix} \begin{pmatrix} (z-a)I_q & 0_q \\ 0_q & I_q \end{pmatrix}. \tag{109}$$

P r o o f. We prove part a). Equality (106) is proved by using (8), (33), (30) and (104). In a similar manner one proves equality (107) by using (9), (105) and (28). Part b) is proved in [8, Corollary 3]. Observe that $Q_{2,n-1}(a)$, $\Theta_{2,n}(a)$, $\Gamma_{1,n}(a)$, $P_{1,n}(a)$ are invertible matrices due to Remark 3 and the fact that $\widehat{H}_{1,n}$, $\widehat{H}_{2,n-1}$, $\widehat{K}_{1,n}$, $\widehat{K}_{2,n}$ are positive definite matrices.

Let us introduce some additional notation:

$$\mathbb{l}_k^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & (z-a)\mathbf{l}_k \end{pmatrix}, \quad \mathbb{m}_k^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & -\mathbf{m}_k \end{pmatrix}, \tag{110}$$

$$\mathbb{l}_k^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & \mathbf{l}_k \end{pmatrix}, \quad \mathbb{m}_k^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & -(z-a)\mathbf{m}_k \end{pmatrix}, \tag{111}$$

$$\mathbb{L}_k^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & \mathbf{L}_k \end{pmatrix}, \quad \mathbb{M}_k^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & -(z-a)\mathbf{M}_k \end{pmatrix}, \tag{112}$$

$$\mathbb{L}_k^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & (z-a)\mathbf{L}_k \end{pmatrix}, \quad \mathbb{M}_k^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & -\mathbf{M}_k \end{pmatrix} \tag{113}$$

and

$$\mathcal{D}_1 := \begin{pmatrix} 0_q & \frac{1}{(b-z)(z-a)} I_q \\ I_q & 0_q \end{pmatrix}, \quad \mathcal{D}_2 := \begin{pmatrix} 0_q & \frac{b-z}{b-a} I_q \\ (b-a)(z-a) I_q & 0_q \end{pmatrix}, \tag{114}$$

$$\mathcal{D}_3 := \begin{pmatrix} 0_q & \frac{1}{b-z} I_q \\ I_q & 0_q \end{pmatrix}, \quad \mathcal{D}_4 := \begin{pmatrix} 0_q & I_q \\ (z-a) I_q & 0_q \end{pmatrix}, \tag{115}$$

$$\mathbf{D}_1 := \begin{pmatrix} (b-z) I_q & 0_q \\ 0_q & I_q \end{pmatrix}, \quad \mathbf{D}_2 := \begin{pmatrix} \frac{1}{z-a} I_q & 0_q \\ 0_q & I_q \end{pmatrix}, \tag{116}$$

$$\mathcal{B}_1^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & Q_{1,n}^*(a) P_{1,n}^{*^{-1}}(a) + \frac{1}{b-a} \Theta_{1,n}^*(a) \Gamma_{1,n}^{*^{-1}}(a) \end{pmatrix}, \tag{117}$$

$$\mathcal{B}_1^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & -\Gamma_{2,n}^*(a) \Theta_{2,n}^{*^{-1}}(a) - (b-a) P_{2,n}^*(a) Q_{2,n}^{*^{-1}}(a) \end{pmatrix}, \tag{118}$$

$$\mathcal{B}_2^{(2n)} := \begin{pmatrix} 0_q & I_q \\ I_q & Q_{2,n-1}^{-1}(a) P_{2,n-1}(a) + \frac{1}{b-a} \Theta_{2,n}^{-1}(a) \Gamma_{2,n}(a) \end{pmatrix}, \tag{119}$$

$$\mathcal{B}_2^{(2n+1)} := \begin{pmatrix} 0_q & I_q \\ I_q & -\Gamma_{1,n}^{-1}(a) \Theta_{1,n}(a) - (b-a) P_{1,n+1}^{-1}(a) Q_{1,n+1}(a) \end{pmatrix}. \tag{120}$$

Lemma 1 *Let the RM* $U^{(2n)}$ *and* $U^{(2n+1)}$ *be defined as in (6) and (7). Furthermore, let* $\mathbb{l}_k^{(2n)}$, $\mathbb{m}_k^{(2n)}$, $\mathbb{l}_k^{(2n+1)}$, $\mathbb{m}_k^{(2n+1)}$, $\mathbb{L}_k^{(2n)}$, $\mathbb{M}_k^{(2n)}$, $\mathbb{L}_k^{(2n+1)}$, $\mathbb{M}_k^{(2n+1)}$, $\mathcal{D}_j$, *for* $j = 1, 2, 3, 4$, $\mathcal{B}_j^{(2n)}$, $\mathcal{B}_j^{(2n+1)}$ *for* $j = 1, 2$ *and* $\mathbf{D}_j$ *for* $j = 1, 2$ *be as in (110)-(113), (114)-(115), (119), (120) and (116), respectively. The identities (1), (2),*

$$U^{(2n)} = \mathbb{M}_0^{(2n)} \mathbb{L}_0^{(2n)} \ldots \mathbb{L}_{n-1}^{(2n)} \mathbb{M}_n^{(2n)} \mathcal{B}_1^{(2n)} \tag{121}$$

and

$$U^{(2n+1)} = \mathbf{D}_2 \mathbb{M}_0^{(2n+1)} \mathbb{L}_0^{(2n+1)} \ldots \mathbb{M}_n^{(2n+1)} \mathbb{L}_n^{(2n+1)} \mathcal{B}_1^{(2n+1)} \mathcal{D}_4 \tag{122}$$

hold.

Proof. We prove (1). By using (114), (110) and (119) clearly the following equalities are valid:

$$\mathcal{D}_1 \mathbb{l}_{-1}^{(2n)} = \begin{pmatrix} \frac{1}{(b-z)(z-a)} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & (z-a)\mathbf{l}_{-1} \\ 0_q & I_q \end{pmatrix}$$

$$\mathbb{m}_k^{(2n)} \mathbb{l}_k^{(2n)} = \begin{pmatrix} I_q & 0_q \\ -\mathbf{m}_k & I_q \end{pmatrix} \begin{pmatrix} I_q & (z-a)\mathbf{l}_k \\ 0_q & I_q \end{pmatrix},$$

$$\mathcal{B}_2^{(2n)} \mathcal{D}_2 = \begin{pmatrix} I_q & 0_q \\ Q_{2,n-1}^{-1}(a) P_{2,n-1}(a) + \frac{1}{b-a} \Theta_{2,n}^{-1}(a) \Gamma_{2,n}(a) & I_q \end{pmatrix} \cdot \begin{pmatrix} (b-a)(z-a) I_q & 0_q \\ 0_q & \frac{b-z}{b-a} I_q \end{pmatrix}.$$

The latter equalities along with (106) imply (1). In a similar manner ones proves (2), (121) and (122).

*

Appendix

In this appendix we reproduce some results from [9] and [8], which are used in the present work.

Definition 11 *[9, Formula (6.2)] Let $H_{2,j}$ be as in (4), and assume that $H_{2,j}$ is a positive definite matrix. Furthermore, let $u_{2,j}$, R_j and v_j be as in (22), (14) and (15). The $2q \times 2q$ matrix polynomial*

$$\widetilde{U}_2^{(2j)}(z,a,b) := \begin{pmatrix} \widetilde{\alpha}_2^{(2j)}(z,a,b) & \widetilde{\beta}_2^{(2j)}(z,a,b) \\ \widetilde{\gamma}_2^{(2j)}(z,a,b) & \widetilde{\delta}_2^{(2j)}(z,a,b) \end{pmatrix}, \; z \in \mathbb{C}, \quad 0 \leq j \leq n-1, \tag{A.1}$$

with

$$\begin{aligned} \widetilde{\alpha}_2^{(2j)}(z,a,b) :=& I_q - (z-a)u_{2,j}^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) v_j, \\ \widetilde{\beta}_2^{(2j)}(z,a,b) :=& (z-a)u_{2,j}^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) u_{2,j}, \\ \widetilde{\gamma}_2^{(2j)}(z,a,b) :=& -(z-a)v_j^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) v_j \end{aligned}$$

and

$$\widetilde{\delta}_2^{(2j)}(z,a,b) := I_q + (z-a)v_j^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a) u_{2,j}$$

is called the second auxiliary matrix of the THMM problem in the case of an odd number of moments.

Remark 9 *[8, Equalities (1.30) and (1.31)] The following identities are valid:*

$$\begin{aligned} \Gamma_{2,j}^*(\bar{z},a)\Theta_{2,j}^{*^{-1}}(a,a) =& -v_j^* R_j^*(\bar{z}) H_{1,j}^{-1} R_j(a) v_j, \\ Q_{1,j+1}^*(\bar{z}) P_{1,j+1}^{*^{-1}}(a) =& -\widetilde{u}_{2,j}^* R_j^*(\bar{z}) K_{2,j}^{-1} R_j(a) \widetilde{u}_{2,j}. \end{aligned}$$

Finally, let us recall the following well-known result below.

Lemma 2 *[1, Proposition 8.2.4] Let $A := \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$ be a Hermitian $(n+m) \times (n+m)$ matrix. Therefore, the following statements are equivalent:*

i) $A > 0$.

ii) $A_{11} > 0$ *and* $A_{12}^* A_{11}^{-1} A_{12} < A_{22}$.

iii) $A_{22} > 0$ *and* $A_{12} A_{22}^{-1} A_{12}^* < A_{11}$.

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