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До Віснику включено статті з математичного аналізу, диференціальних рівнянь, математичної теорії керування та механіки, які містять нові теоретичні результати у зазначених галузях і мають прикладне значення.

Для викладачів, наукових працівників, аспірантів, працюючих у відповідних або суміжних сферах.

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Статті пройшли внутрішнє та зовнішнє рецензування.

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Bishop-Phelps-Bollobás modulus of a uniformly non-square Banach space

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Chica, Kadets, Martín and Soloviova demonstrated recently that the Bishop-Phelps-Bollobás modulus Φ_X^S of a Banach spaces X can be estimated from above through the parameter of uniform non-squareness $\alpha(X)$: $\Phi_X^S(\varepsilon) \leq \sqrt{2\varepsilon}\sqrt{1-\frac{1}{3}\alpha(X)}$. In this short note we demonstrate that the right-hand side in the above theorem cannot be substituted by anything smaller than $\sqrt{2\varepsilon}\sqrt{1-\alpha(X)}$.
Keywords: Bishop-Phelps theorem; uniformly non-square spaces.

Соловйова М. В. **Модулі Бішопа-Фелпса-Болобаша в рівномірно неквадратних банахових просторах.** Чіка, Кадець, Мартін, Соловйова нещодавно довели, що модуль Бішопа-Фелпса-Болобаша Φ_X^S банахового простора X може бути оцінений зверху через параметр рівномірної неквадратності $\alpha(X)$: $\Phi_X^S(\varepsilon) \leq \sqrt{2\varepsilon}\sqrt{1-\frac{1}{3}\alpha(X)}$. У цій короткій статті ми покажемо, що права частина оцінки не може бути змінена на щось меньше, ніж $\sqrt{2\varepsilon}\sqrt{1-\alpha(X)}$.
Ключові слова: теорема Бішопа-Фелпса, рівномірно неквадратні простори.

Соловьева М. В. **Модули Бишопа-Фелпса-Боллобаша в равномерно неквадратных банаховых пространствах** Чика, Кадец, Мартин, Соловьёва недавно доказали, что модуль Бишопа-Фелпса-Боллобаша Φ_X^S банахового пространства X может быть оценен сверху через параметр равномерной неквадратности $\alpha(X)$: $\Phi_X^S(\varepsilon) \leq \sqrt{2\varepsilon}\sqrt{1-\frac{1}{3}\alpha(X)}$. В этой короткой статье мы покажем, что правая часть этой оценки не может быть заменена на что-то меньшее, чем $\sqrt{2\varepsilon}\sqrt{1-\alpha(X)}$.
Ключевые слова: теорема Бишопа-Фелпса, равномерно неквадратные пространства.
2000 Mathematics Subject Classification 46B04, 46B20.

Introduction

In this paper letter X stands for a real Banach space. A functional $x^* \in X^*$ *attains its norm*, if there is an $x \in S_X$ with $x^*(x) = \|x^*\|$. The classical Bishop-Phelps theorem states that the set of norm attaining functionals on a Banach space is norm dense in the dual space ([1], see also [6, Chapter 1]). A refinement of this theorem, nowadays known as the Bishop-Phelps-Bollobás theorem [2], was proved by B. Bollobás and allows to approximate at the same time a functional and a vector in which it almost attains the norm. Very recently, the following quantity have been introduced [4] which measure, for a given Banach space, what is the best possible Bishop-Phelps-Bollobás theorem in this space. Denote by S_X and B_X the unit sphere and the closed unit ball of X respectively. We will also use the notation

$$\Pi(X) := \{(x, x^*) \in X \times X^* : \|x\| = \|x^*\| = x^*(x) = 1\}.$$

Definition 1 (Bishop-Phelps-Bollobás modulus, [4]) *Let X be a real Banach space. The* spherical Bishop-Phelps-Bollobás modulus *of the space X is the function $\Phi_X^S : (0, 2) \longrightarrow \mathbb{R}^+$ such that given $\varepsilon \in (0, 2)$, $\Phi_X^S(\varepsilon)$ is the infimum of those $\delta > 0$ satisfying that for every $(x, x^*) \in S_X \times S_{X^*}$ with $x^*(x) > 1 - \varepsilon$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \delta$ and $\|x^* - y^*\| < \delta$.*

It is known (see, for example, [4, Theorem 2.1]) that for every Banach space X and every $\varepsilon \in (0, 2)$ one has $\Phi_X^S(\varepsilon) \leq \sqrt{2\varepsilon}$. This estimate is sharp for the two-dimensional real space $\ell_1^{(2)}$ (see [2] or [4, Example 2.5]).

Uniformly non-square spaces were introduced by James [7] as those spaces whose two-dimensional subspaces are uniformly separated from $\ell_1^{(2)}$. The main result of [7] – the reflexivity of uniformly non-square spaces – was the origin of the theory of superreflexive spaces.

Recall that a Banach space X is *uniformly non-square* if and only if there is $\alpha > 0$ such that

$$\frac{1}{2}(\|x + y\| + \|x - y\|) \leq 2 - \alpha$$

for all $x, y \in B_X$. The *parameter of uniform non-squareness* of X, which we denote $\alpha(X)$, is the best possible value of α in the above inequality. In other words,

$$\alpha(X) := 2 - \sup_{x,y \in B_X} \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) \right\}.$$

With this notation X is uniformly non-square if and only if $\alpha(X) > 0$. In a uniformly non-square space the estimate $\Phi_X^S(\varepsilon) \leq \sqrt{2\varepsilon}$ can be improved.

Theorem 1 (Theorem 3.3 of [5]) *Let X be a Banach space with $\alpha(X) > 0$. Then,*

$$\Phi_X^S(\varepsilon) \leq \sqrt{2\varepsilon}\sqrt{1 - \frac{1}{3}\alpha(X)} \quad \textit{for} \quad 0 < \varepsilon < \frac{1}{2} - \frac{1}{6}\alpha(X).$$

Although we don't know whether the above estimate of $\Phi_X^S(\varepsilon)$ through $\alpha(X)$ is sharp, we are able to demonstrate (and this is the goal of this short article) that this result cannot be improved too much. Namely, we demonstrate that the unknown optimal estimate of $\Phi_X^S(\varepsilon)$ through $\alpha(X)$ cannot be better than $\sqrt{2\varepsilon}\sqrt{1-\alpha(X)}$.

The main result

We will make a use of "hexagonal spaces" X_ρ introduced in [8] and the description of $\Pi(X_\rho)$ from that paper. Fix a $\rho > \frac{1}{2}$ and denote X_ρ the linear space $\mathbb{R}^2$ equipped with the norm

$$\|(x_1, x_2)\| = \|(x_1, x_2)\|_\rho = \max\left\{|x_1 - \frac{1-\rho}{\rho}x_2|, |x_2 - \frac{1-\rho}{\rho}x_1|, |x_1 + x_2|\right\}.$$

In other words,

$$\|(x_1, x_2)\| = \begin{cases} |x_1 + x_2|, & \text{if } x_1x_2 \geq 0; \\ |x_1 - \frac{1-\rho}{\rho}x_2|, & \text{if } x_1x_2 < 0 \text{ and } |x_1| > |x_2|; \\ |x_2 - \frac{1-\rho}{\rho}x_1|, & \text{if } x_1x_2 < 0 \text{ and } |x_1| \leq |x_2|. \end{cases}$$

and the unit ball B_ρ of X_ρ is the hexagon $abcdef$, where $a = (1, 0); b = (0, 1); c = (-\rho, \rho); d = (-1, 0); e = (0, -1)$; and $f = (\rho, -\rho)$.

The dual space to X_ρ is $\mathbb{R}^2$ equipped with the polar to B_ρ as its unit ball. So the norm on X_ρ^* is given by the formula

$$\|(x_1, x_2)\|^* = \|(x_1, x_2)\|_\rho^* = \max\{|x_1|, |x_2|, \rho|x_1 - x_2|\},$$

and the unit ball B_ρ^* of X_ρ^* is the hexagon $a^*b^*c^*d^*e^*f^*$, where $a^* = (1, 1)$; $b^* = \left(-\frac{1-\rho}{\rho}, 1\right)$; $c^* = \left(-1, \frac{1-\rho}{\rho}\right)$; $d^* = (-1, -1)$; $e^* = \left(\frac{1-\rho}{\rho}, -1\right)$; and $f^* = \left(1, -\frac{1-\rho}{\rho}\right)$. The corresponding spheres S_ρ and S_ρ^* are shown on Fig. 1 and 2 respectively.

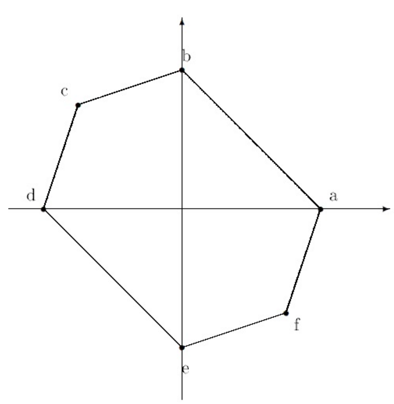

Fig. 1: Unit sphere of X_ρ.

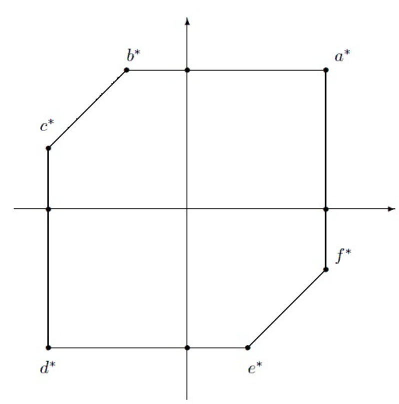

Fig. 2: Unit sphere of X_ρ^*.

In the case of $\rho = \frac{1}{2}$ the sphere of X_ρ reduces to the square $abde$, and consequently $X_{1/2}$ is isometric to the spaces $\ell_1^{(2)}$ and $\ell_\infty^{(2)}$. When $\rho > \frac{1}{2}$, the space X_ρ is not isometric to $\ell_\infty^{(2)}$. Let us calculate the parameter of uniform non-squareness for X_ρ.

Lemma 1 *Let $\rho \in [1/2, 1]$. Then, in the space $X = X_\rho$,*

$$\alpha(X_\rho) = 1 - \frac{1}{2\rho}. \tag{1}$$

Proof. Consider $\varphi(x,y) = \frac{1}{2}(\|x+y\| + \|x-y\|)$. Then $\alpha(X) = 2 - \sup\{\varphi(x,y) : (x,y) \in B_{X_\rho} \times B_{X_\rho}\}$. Since $\varphi : B_{X_\rho} \times B_{X_\rho} \to \mathbb{R}$ is a convex function, it attains its maximum at some extreme point of $S_{X_\rho} \times S_{X_\rho}$, i.e. at a point of the form (x,y) with $x, y \in \{a,b,c,d,e,f\}$. Also, $\varphi(x,y) = \varphi(y,x) = \varphi(x,-y)$, so by symmetry of the function and symmetry of the ball, is sufficient to check values of functions φ for the following two pairs (x,y): $x = a, y = b$ and $x = a, y = c$.

If $x = a = (1,0), y = b = (0,1)$, then $\|x+y\| = \|(1,1)\| = 2$, $\|x-y\| = \|(1,-1)\| = 1 + \frac{1-\rho}{\rho} = \frac{1}{\rho}$. So, $\varphi(a,b) = 1 + \frac{1}{2\rho}$.

If $x = a = (1,0), y = c = (-\rho,\rho)$, then $\|x+y\| = \|(1-\rho,\rho)\| = 1-\rho+\rho = 1$, $\|x-y\| = \|(1+\rho,-\rho)\| = 1+\rho+1-\rho = 2$. So, $\varphi(a,c) = 1 + \frac{1}{2} \le 1 + \frac{1}{2\rho}$.

Therefore $\max\{\varphi(x,y) : (x,y) \in B_{X_\rho} \times B_{X_\rho}\} = 1 + \frac{1}{2\rho}$, and consequently $\alpha(X_\rho) = 1 - \frac{1}{2\rho}$. The lemma is proved.

The set $\Pi(X_\rho)$ is the following polygon in $\mathbb{R}^2 \times \mathbb{R}^2$:
$\Pi(X_\rho) = \{(a,x^*) : x^* \in [f^*,a^*]\} \cup \{(x,a^*) : x \in [a,b]\} \cup \{(b,x^*) : x^* \in [a^*,b^*]\}$
$\cup\{(x,b^*) : x \in [b,c]\} \cup \{(c,x^*) : x^* \in [b^*,c^*]\} \cup \{(x,c^*) : x \in [c,d]\}$
$\cup\{(d,x^*) : x^* \in [c^*,d^*]\} \cup \{(x,d^*) : x \in [d,e]\} \cup \{(e,x^*) : x^* \in [d^*,e^*]\}$
$\cup\{(x,e^*) : x \in [e,f]\} \cup \{(f,x^*) : x^* \in [e^*,f^*]\} \cup \{(x,f^*) : x \in [f,a]\}$,
where we use brackets like $[\cdot,\cdot]$, $[\cdot,\cdot[$ to denote line segments in a linear space, for example, $[a,b] = \{\lambda b + (1-\lambda)a : 0 \le \lambda \le 1\}$; and parenthesis $(\cdot,\cdot)$ are reserved to denote an element of a Cartesian product.

Theorem 2 *For every $\alpha \in [0,1/2]$ there is a Banach space X with $\alpha(X) = \alpha$ such that*

$$\Phi_X^S(\varepsilon) \ge \sqrt{2\varepsilon}\sqrt{1-\alpha(X)} \tag{2}$$

for all $0 < \varepsilon < 1$.

Proof. Let us demonstrate that the space $X = X_\rho$ with $\rho = \frac{1}{2(1-\alpha)}$ is what we are looking for. The direct application of lemma 1 gives $\alpha(X) = \alpha$, so what remains to show is (2).

Denote $x = (1 - \sqrt{\varepsilon\rho}, \sqrt{\varepsilon\rho}), x^* = (1, 1 - \sqrt{\varepsilon/\rho})$. Then, $x \in]a,b[$, $x^* \in]a^*,f^*[$ and $x^*(x) = 1 - \varepsilon$. In order to demonstrate (2) it is sufficient to prove the absence of such a pair $(y,y^*) \in \Pi(X)$ that $\max\{\|x-y\|, \|x^*-y^*\|\} < \sqrt{2\varepsilon}\sqrt{1-\alpha}$.

Denote $r = \sqrt{2\varepsilon}\sqrt{1-\alpha}$ and consider the set U of those $y \in S_X$ that $\|x-y\| < r$. U is the intersection of S_X with the open ball of radius r centered in x (U is the bold line in Fig. 3). The radius of the ball equals to the distance from x to a:

$$\|x-a\| = \|(-\sqrt{\varepsilon\rho}, \sqrt{\varepsilon\rho})\| = \sqrt{\varepsilon\rho} + \frac{1-\rho}{\rho}\sqrt{\varepsilon\rho} = \sqrt{\varepsilon/\rho} = \sqrt{2\varepsilon}\sqrt{1-\alpha} = r,$$

which explains the picture for small r. Also for bigger values of r the set U can contain points b and c, but it never contains any point of $[d,e], [e,f]$ and $[f,a]$. Observe that the open ball of radius $1/\rho$ centered in b contains the set U, as if $h \in U$, we have $\|b-h\| \le \|b-x\| + \|x-h\| < \|b-x\| + \|x-a\| = \|b-a\| = 1/\rho$. Therefore it is sufficient to check that the distance from b to every point of $[d,e], [e,f]$ and $[f,a]$ is no less than $1/\rho$. Indeed, if $s = (-w, w-1)$ is a point of $[d,e]$ $(0 \le w \le 1)$, then

$$\|b-s\| = \|(w, 2-w)\| = w + 2 - w = 2 \ge 1/\rho.$$

If $s = (w, \frac{1-\rho}{\rho}w - 1)$ is a point of $[e,f]$, $0 \le w \le \rho$, and so

$$\|b-s\| = \|(-w, 1 - \frac{1-\rho}{\rho}w + 1)\| = \frac{1-\rho}{\rho}w + 2 - \frac{1-\rho}{\rho}w = 2 \ge 1/\rho.$$

If s is a point of $[f,a]$, $\rho \le w \le 1$, we shall consider cases $\rho < 1$ and $\rho = 1$ separately. For $\rho < 1$ we have $s = (w, -\frac{\rho}{1-\rho}(1-w))$, then

$$\|b-s\| = \|(-w, 1 + \frac{\rho}{1-\rho}(1-w))\| = \frac{\rho}{1-\rho}w + 1 + \frac{\rho}{1-\rho} - \frac{\rho}{1-\rho}w \ge 2 \ge 1/\rho.$$

And for $\rho = 1$ we have $s = (1, -w)$, $0 \le w \le 1$. Hence

$$\|b-s\| = \|(-1, 1+w)\| = max\{1, 1+w\} \ge 1 = 1/\rho.$$

So, $U \subset]a,b] \cup [b,c] \cup [c,d[$.

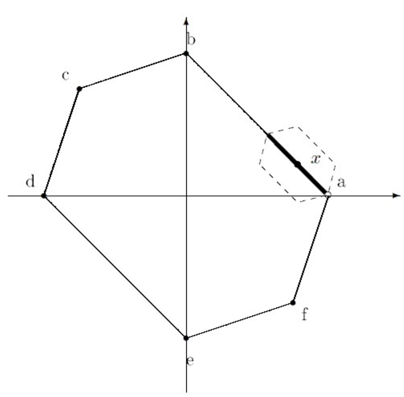

Fig. 3: The set U.

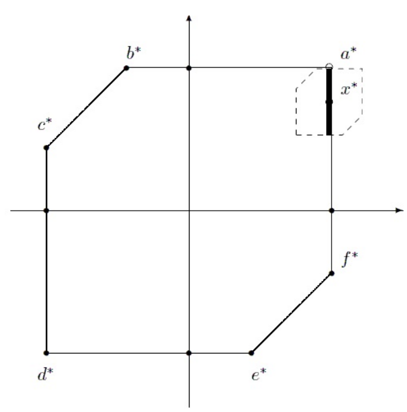

Fig. 4: The set V.

Consider also the set V of those $y^* \in S_{X^*}$ that $\|x^* - y^*\| < r$. V is the intersection of S_{X^*} with the open ball of radius r centered in x^* (the bold line in Fig. 4). The radius of the ball equals to the distance from x^* to a^*: $\|x^* - a^*\| = \|(0, -\sqrt{\varepsilon/\rho})\| = \sqrt{\varepsilon/\rho} = r$.

What remains to show is that $(y, y^*) \notin \Pi(X)$ for every $y \in U$ and every $y^* \in V$. The latter fact follows immediately form the above descriptions of the sets $\Pi(X_\rho)$ and U together with the fact that $V \subset]d^*, e^*] \cup [e^*, f^*] \cup [f^*, a^*[$.

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The field of vertical electric dipole placed above the spiral conductive unclosed sphere

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The problem of the electromagnetic field an open spiral conductive sphere is analyzing. The method of regularization of operator tasks is applied. The integral equations with a weak singularity in the kernel is used. The infinite system of algebraic equations of type II with a compact operator in ℓ_2 is received. Some properties of electromagnetic fields are studied.
Keywords: spiral conductive sphere, vertical dipole, compact operator.

Резуненко В.О. **Поле вертикального електричного диполя, який розміщений над спірально провідною незамкненою сферою.** Досліджується задача про електромагнітне поле спірально провідної незамкненої сфери. Застосовано метод регуляризації оператора задачі, вмкористовано розв'язки інтегральних рівняннь із слабкою особливістю у ядрі. Одержано нескінченну систему алгебраїчних рівнянь II роду з компактним оператором у ℓ_2. Вивчені деякі властивості електромагнітних полів.
Ключові слова: спірально провідна сфера, вертикальний диполь, компактний оператор.

Резуненко В.А. **Поле вертикального электрического диполя, размещённого над спирально проводящей незамкнутой сферой.** Исследуется задача об электромагнитном поле спирально проводящей незамкнутой сферы. Применены метод регуляризации оператора задачи, интегральные уравнения со слабой особенностью в ядре. Получена бесконечная система алгебраических уравнений II рода с компактным оператором в ℓ_2. Изучены некоторые свойства электромагнитных полей.
Ключевые слова: спирально проводящая сфера, вертикальный диполь, компактный оператор.

2010 Mathematics Subject Classification: 65N12; 35A25; 78A455.

1. Introduction

The methods of regularization of matrix and integral operators of applied problems occupy a prominent place among the numerical-analytical methods [1], [2]. The variant of methods [1], [2] is used for analysis of the electrodynamic properties of the unclosed spiral conductive spherical surface. The spiral antennas and devices have small size and the lightweight. They are power saving ones. They allow to control the polarization of the radiation fields. The spiral antennas have been successfully used on mobile objects to communicate at short, medium and at very long distances [3]-[5]. We note that there are many experimental papers on this subject. The number of theoretical works is comparably small. The purpose of our work is the construction of a numerical analytical algorithm for study of fields properties of the spiral conductive unclosed sphere [1], [2]. The spiral conductive unclosed sphere is irradiated by the vertical electric dipole field. The dipole is placed above the sphere with a circular aperture on its axis of symmetry. We apply the method of regularization of problem's operator. We use the solutions of integral equations with a weak singularity in the kernels. The main part of the matrix operator is extracted and inverted. The infinite linear algebraic system of second kind with compact operator in Hilbert space l_2 is obtained. The limit cases of formulation of the problem and properties of solutions are considered.

2. Formulation of the problem

The origin of Cartesian and spherical systems of coordinates are placed in the geometrical center of the sphere of radius $r = a$. Let us cut the sphere by a horizontal plane into two parts. Consider its upper part as an unclosed sphere with a circular aperture. Let the polar angle θ of the edge of the aperture be equal θ_0. The polar angle θ on the aperture is changing from θ_0 to π. Let the vertical electrical dipole be placed on the axis of symmetry of the unclosed sphere on the axis OZ at the point $z = b > a$. We assume that the surface of an unclosed sphere is infinitely thin and spiral conductive. Let β will be the angle between the lines of conductivity of the electric current on the unclosed sphere and the lines of the meridians on the sphere. The sphere conducts the current in selected directions only. We note that the line of the conductivity on the sphere may be represented as follows: $x = sin(\eta)cos(14\eta)$, $y = sin(\eta)sin(14\eta)$, $z = 1 + cos(\eta)$, where η is a dimensionless parameter, which varies in bands $[0, \pi/2]$ (fig.1).The dipole field $\vec{E}^{(0)}$, $\vec{H}^{(0)}$ meets an unclosed sphere and creates secondary electromagnetic fields: $\vec{E}^{(1)}$, $\vec{H}^{(1)}$ in the area $0 \leqslant r < a$ and $\vec{E}^{(2)}$, $\vec{H}^{(2)}$in the area $r > a$. By definition, the total field in the area $0 \leqslant r < a$ is equal to $\vec{E}^{(1)}$, $\vec{H}^{(1)}$. According to the superposition principle of electromagnetic fields, the total field in the area $r > a$ is the sum of fields $\vec{E}^{(0)} + \vec{E}^{(2)}$ and $\vec{H}^{(0)} + \vec{H}^{(2)}$. The time dependence of the fields is taken as $exp(-i\omega t)$, where ω is the angular frequency, $\omega = 2\pi/\lambda$, λ is a wavelength of the dipole field.

The total electromagnetic fields outside of the unclosed sphere satisfy the

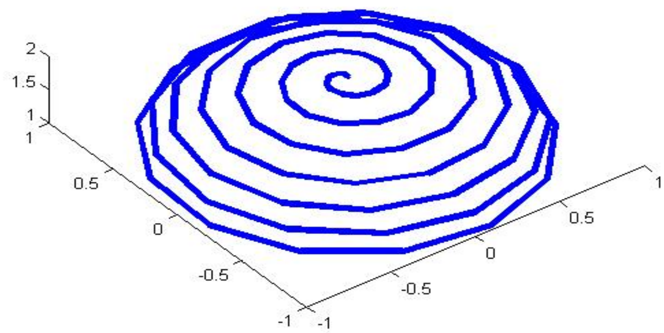

Fig.1: The line of spiral conductivity of the electric current.

following conditions: 1) the Maxwell and material equations:

$$rot\,\vec{E} = ik\vec{H}, \quad rot\,\vec{H} = -ik\vec{E}, \quad div\,\vec{D} = \rho, \quad div\,\vec{B} = 0, \tag{1}$$

$$\vec{D} = \varepsilon\vec{E}, \quad \vec{B} = \mu\vec{H}, \quad \vec{J} = \sigma\vec{E};$$

where $k = \omega\sqrt{\epsilon\mu}c^{-1}$, ϵ, μ and σ are the dielectric permittivity, the magnetic permeability and the conductivity of the medium, ρ is the charge density, c is the speed of light in vacuum; 2) the energy boundedness in any restricted volume A in R^3:

$$\int_A \left(\varepsilon|\vec{E}|^2 + \mu|\vec{H}|^2\right)\,dx\,dy\,dz < \infty, \tag{2}$$

where the volume A may contain the edge of the unclosed sphere; 3) the condition of fields radiation on infinity:

$$\lim r[\frac{\partial\Psi}{\partial r} - ik\Psi] = 0, r \to \infty,$$

where Ψ is any component $\vec{E}$ or $\vec{H}$.

3. Boundary conditions

In addition to conditions 1)-3), the total fields satisfy the boundary conditions. We write the conditions for the field's components

$$\vec{E}(E_r, E_\theta, E_\varphi), \quad \vec{H}(H_r, H_\theta, H_\varphi). \tag{3}$$

in the spherical coordinate system.

B1) the field's components on the surface of the unclosed sphere $\{r = a, 0 \le \theta < \theta_0, \phi \in [0, 2\pi]\}$ satisfy the conditions of the spiral conductivity:

$$(H_\theta^{(2)} - H_\theta^{(1)}) + (H_\varphi^{(2)} + H_\varphi^{(0)} - H_\varphi^{(1)})\, tg\,\beta = 0,$$
$$E_\theta^{(0)} + E_\theta^{(2)} = E_\theta^{(1)}, \quad E_\varphi^{(2)} = E_\varphi^{(1)},$$
$$E_\theta^{(1)} - E_\theta^{(1)}\, tg\,\beta = 0; \tag{4}$$

B2) the total fields are continuous on the aperture of the unclosed sphere $\{r = a, \theta_0 < \theta \leq \pi, \phi \in [0, 2\pi]\}$:

$$\vec{E}^{(2)} + \vec{E}^{(0)} = \vec{E}^{(1)}, \quad \vec{H}^{(2)} + \vec{H}^{(0)} = \vec{H}^{(1)}. \tag{5}$$

The total fields satisfy the requirement for singularity in the dipole placement point. The problem (1) - (5) has a unique solution [8].

To solve the problem (1) - (5), we use the methods of regularization of the auxiliary integral and matrix operators. First, using the Debye u electric and v magnetic scalar potentials, the components of the field (3) are written. The fields components are uniquely expressed by the Debye potentials. The scalar potentials u, v satisfy the Helmholtz equation, which follows from the Maxwell equations (1), in particular, $\Delta u + k^2 u = 0$. We write the Helmholtz equation in the spherical coordinate system and separate the variables in the equation. The potentials are represented by the Fourier series. We note that the magnetic potential of the vertical electric dipole, placed on the axis OZ, is equal to zero: $v^{(0)} = 0$. The electric potential of the dipole is present by the series of eigenfunctions of the auxiliary the Sturm-Liouville problem as follows

$$u^{(0)} = \sum_{n=1}^{\infty} F(n) \frac{P_n(\cos\theta)}{k^3 r b^2} \left\{ \begin{array}{ll} \psi_n(kr)\xi_n(kb), & r < b, \\ \psi_n(kb)\xi_n(kr), & r > b, \end{array} \right. \quad F(n) = 2n + 1. \tag{6}$$

We note that the modulus of dipole moment $\vec{P}$, which is directed along the axis OZ is equal to unity in the expression (6). We also take into account that the dipole is placed in the upper half space on the axis OZ above the unclosed sphere. In (6) $\psi_n(x), \xi_n(x)$ are spherical Bessel and Hankel functions in the Debye's notation of the first and 3-th kinds, respectively, of the n-th order of argument x; $P_n(\cos\theta)$ are Legendre polynomials of the first kind of the n-th power and zero order of the argument $\cos\theta$. We look for the secondary potentials (7), (8) in the form of series (6):

$$\left. \begin{array}{l} u^{(1)} \\ \nu^{(1)} \end{array} \right\} = \sum_{n=1}^{\infty} F(n) \frac{P_n(\cos\theta)}{kr} \left\{ \begin{array}{ll} A_n \psi_n(kr), & r < a, \\ B_n \xi_n(kr), & r > a, \end{array} \right. \tag{7}$$

$$\left. \begin{array}{l} u^{(2)} \\ \nu^{(2)} \end{array} \right\} = \sum_{n=1}^{\infty} F(n) \frac{P_n(\cos\theta)}{kr} \left\{ \begin{array}{ll} C_n \psi_n(kr), & r < a, \\ D_n \xi_n(kr), & r > a. \end{array} \right. \tag{8}$$

Here in (7), (8) we take into account the occurrence of magnetic potentials in the secondary fields, which are scattered by the spiral conductive unclosed sphere. The unknown coefficients A_n, B_n, C_n, D_n of the series (7) and (8) belong to the Hilbert space (see (2)) with certain weights, which are different for different coefficients.

4. The paired functional equations containing the associated Legendre functions

Using the boundary conditions (4), (5), we get the three linear equations of connection for four unknown coefficients A_n, B_n, C_n, D_n for each $n = 1, 2, 3, ...$. We used here the orthogonality of the associated Legendre functions of the first kind of the first order with the weight $sin\theta$ on the segment $[0, \pi]$. The coefficients A_n, B_n and D_n are expressed in terms of the coefficients C_n by the three equations of connection. To find coefficients C_n we deduce the paired functional equations. We use all the boundary conditions (4), (5) for all the components of the unknown fields $\vec{E}^{(1)}, \vec{H}^{(1)}, \vec{E}^{(2)}, \vec{H}^{(2)}$ from (3). As a result, we obtain the system of paired functional equations, which allows to find the coefficients (8) of potential $u^{(2)}$:

$$\sum_{n=1}^{\infty} C_n F(n) \frac{1}{\psi_n'(ka)} P_n^1(\cos\theta) = 0, \qquad \theta_0 < \theta \le \pi, \tag{9}$$

$$\sum_{n=1}^{\infty} C_n \frac{F(n)}{\psi_n'(ka)} \left\{ (tg\,\beta)^2 \psi_n(ka)\xi_n(ka) + \psi_n'(ka)\xi_n'(ka) \right\} P_n^1(\cos\theta) =$$

$$-(kb)^{-2} \sum_{n=1}^{\infty} F(n)\psi_n'(ka)\xi_n(kb)\psi_n(kb)P_n^1(\cos\theta), \qquad 0 \le \theta < \theta_0, \tag{10}$$

where the prime of the functions $\psi_n(\cdot), \xi_n(\cdot)$ means the differentiation with respect to the argument. To find all coefficients of the potential (7), (8) there is only one paired system of functional equations. In contrast to [11], the questions of the division of polarization fields and search for additional constants of integration do not arise. The system of paired functional equations (9)-(10) is of the first kind with complicated kernels, which involve various spherical functions. The multipliers of the unknown coefficients C_n in (9) and (10) are different and have different rates of decrease as $n \to \infty$. Even taking into account the orthogonality of the associated Legendre functions with the weight $sin\theta$ in $L_2(0, \pi)$, such systems can not be solved analytically. The systems of this type appear in many problems of fields diffraction on open structures. There are many direct numerical methods developed for their approximate solution. These methods are more general than the analytical ones. However, such methods do not allow to evaluate the accuracy of the solutions. This fact is important in the analysis, e.g., resonance oscillations of the investigated fields. In addition, the direct numerical methods also require the use of considerable computing resources. We apply analytical method for the regularization of the system (9), (10) [7,9-15,19,20]. As a result, we obtain the infinite system of linear algebraic equations of the second kind, which is successfully solved numerically and analytically.

5. The infinite system of linear algebraic equations of the second kind

We transform the system (9), (10) into an equivalent system of functional equations, which include the trigonometric functions instead of Legendre functions. For this purpose we use the convergence of the series (10) in $L_2(0,\pi)$ and integrate the equation (10) term by term. Here we use the equality $P_n^1(\cos\theta) = -[P_n(\cos\theta)]'$. Here the constant $T^{(0)}$ of integration arises. We find the constant $T^{(0)}$ below in (16). The Meller-Dirichlet integral representation for the Legendre polynomials (11)

$$P_n(\cos\theta) = \pi^{-1}\sqrt{2}\int_0^\theta (\cos\phi - \cos\theta)^{-0.5}\cos(n+0.5)\phi\, d\phi \tag{11}$$

is substituted into the integrated equation (10). Then the integral representation of the type Meller - Dirichlet for the associated Legendre functions (12)

$$P_n^1(\cos\theta) = [\pi sin\theta]^{-1}\frac{n(n+1)}{2n+1}\sqrt{2}\int_\theta^\pi (\cos\theta - \cos\phi)^{-0.5}\cos(n+0.5)\phi\cdot\sin\phi\, d\phi \tag{12}$$

is substituted into equation (9). Using the convergence of the series in $L_2(0,\pi)$, the order of integration and summation in both equations (9) and (10) changes. In this case we have two integral equations of the first kind with the weak singularities in the kernels. The singularities are due to the presence of radicals in (11) and (12). So, we get from (10) the integral equation $\int_0^\theta (\cos\phi - \cos\theta)^{-0.5} f_1(\phi)\, d\phi = 0$, where

$$f_1(\phi) = \sum_{n=1}^\infty F(n)\left\{C_n[\psi_n'(ka)]^{(-1)}[(tg\,\beta)^2\psi_n(ka)\xi_n(ka) + \psi_n'(ka)\xi_n'(ka)] - \right.$$

$$\left. \psi_n'(ka)\xi_n(kb)\,\psi_n(kb)/(kb)^2\right\}\cdot\cos(n+0.5)\phi - T^{(0)}\cos(0.5)\phi.$$

The solution of the integral equation is found in $L_2(0,\pi)$ by using the composition with the kernel of the equation [6, 7, 15]. We receive the unique trivial solution: $f_1(\phi) = 0, \phi \in (0,\theta_0)$.

Similarly, from the equation (9), we obtain the integral equation $\int_\theta^\pi (\cos\theta - \cos\phi)^{-0.5} f_2(\phi)\, d\phi = 0$, where $f_2(\phi)$ is represented by the trigonometric Fourier series. That integral equation also has the unique trivial solution in $L_2(0,\pi)$: $f_2(\phi) = 0, \phi \in (\theta_0,\pi)$. We receive a new system of functional equations of the first kind. Next, we transform the system of the first kind into the system of the second kind.

For this purpose we apply the methods [1,2,7,9-15,19,20] and use the properties of the Bessel and the Hankel functions [21]. Then we do some linear transformations of the system of functional equations and find the main part of the system. Next, we relabel the coefficients C_n to the new coefficients y_n (13) and introduce the small parameters ε_n (14):

$$y_n = C_n F(n) n(n+1)\left[\psi_n^1(ka)(2n+1)\right]^{-1}, \tag{13}$$

$$\varepsilon_n = 1 + ika\frac{2n+1}{(n(n+1))}\{\psi_n'(ka)\xi_n'(ka) + (tg\,\beta)^2\psi_n(ka)\xi_n(ka)\}. \qquad (14)$$

Now we inverse analytically the main part of the functional equations of the second kind. For this, the methods [1,2,7,9–15, 19, 20] and the method of discrete Fourier transform are used. As a result, we obtain the infinite system of linear algebraic equations of the second kind:

$$\begin{aligned} y_n = (\pi)^{-1}\sum_{m=1}^{\infty} y_m\varepsilon_m q_{n,m}(\theta_0) - \frac{ika}{\pi}T^{(0)}q_{n,0}(\theta_0) - \\ \frac{ia}{k\pi b^2}\sum_{m=1}^{\infty} F(m)\psi_m^1(ka)\xi_m(kb)q_{n,m}(\theta_0). \end{aligned} \qquad (15)$$

Also, we find the integration constant $T^{(0)}$ for the equation (9) and substitute it in (14) as :

$$T^{(0)} = \frac{i}{ka}\sum_{m=1}^{\infty} y_m\varepsilon_m \left\{1 + ika\,\psi_m^1(ka)\,\xi_m(kb)F(m)\cdot(kb)^{-2}\right\}\frac{q_{m,0}(\theta_0)}{q_{0,0}(\theta_0)}, \qquad (16)$$

where

$$q_{n,m}(\theta_0) = 2\int_0^{\theta_0}[\cos(n+0.5)\phi][cos(m+0.5)\phi]d\phi. \qquad (17)$$

Consider the properties of the resulting system (15). For any values ka and $\beta \in [0, \frac{\pi}{2})$ the small parameter (14) vanishes comparably quickly, proportionally to n^{-2}, when $n \to \infty$. The auxiliary values $q_{n,m}(\theta_0)$ in (17) are uniformly bounded by 2π for any $n, m \geq 1$ and any $\theta_0 \in [0, \pi]$. In addition, the values $q_{n,m}(\theta_0)$ for fixed $n = n_0$ vanish proportionally to n^{-1}, when $m \to \infty$. Similarly, the values $q_{n,m}(\theta_0)$ for fixed $m = m_0$ vanish, proportionally to n^{-1}, when $n \to \infty$. The matrix elements $\{G_{n,m}\}_{n,m=1}^{\infty}$ of the system $Y = GY + Q$ (15) for fixed $n = n_0$ vanish when $m \to \infty$ and they vanish for fixed $m = m_0$, when $n \to \infty$. The eigenvalues of the system's matrix operator differ from the unity. The right column of the system (15) belongs to l_2. The system (15) has the compact matrix operator in l_2 and a unique solution in l_2. It is solved numerically for arbitrary geometric and frequencies parameters of the problem. The system is solved analytically, in particular, by the method of successive approximations for the large apertures in the sphere ($0 \leq \theta_0 \ll 1$). This follows from the fact that the norm in l_2 of the matrix G of the system (15) is proportional to θ_0 for small θ_0. This method can be applied successfully for small apertures in the sphere ($0 \leq \pi - \theta_0 \ll 1$) after simple linear transformations of the system (15).

6. Conclusions

1. The system (15) is constructed for the study of electromagnetic fields in the case of placing of an electrical dipole in the point $z = b > a$ on the axis OZ.

The system (15) can be modified for the case of the dipole placed in the point $z = -b$ on the axis OZ. For this it is necessary to relabel the coefficients F_n in (6)-(8) as follows: $F_n^{(1)} = (-1)^{n+1}F(n), n \geq 1$.
2. Introducing the new coefficients $y_n^{(1)} = y_n n^{-2}, n \geq 1$, instead of the coefficients y_n (13), the speed of convergence of the analytical and numerical methods for solving the system (15) can be increased.
3. The polarization of the electromagnetic field of structure varies non monotonically from a linear to elliptical and almost circular with a change of the angle β of the spiral conductivity of the sphere and with an increase of the angle θ_0 between zero and π.
4. The reduced resonant frequencies $\chi_{n,m}, n, m > 1$ of the structure for small $\theta_1 = \pi - \theta_0 \ll 1$ and small β differ from the ones of the closed sphere $\chi_{n,m}^{(0)}$ on the coefficient, which is proportional to θ_1: $\chi_{n,m} = \chi_{n,m}^{(0)} + O(\theta_1)$, when $\theta_1 \to 0$ [10,16-20].
5. The sphere disappears completely when $\theta_0 \to 0$ and it turns into a closed spiral conductive sphere when $\theta_0 \to \pi$. The unclosed sphere becomes almost perfectly conductive, if β decreases from $\pi/2$ to zero. The electromagnetic field penetrates almost completely through the spiral conductive sphere when $\beta \to \pi/2$.
6. The constructed numerical-analytical algorithm can be generalized, for example, to calculate the electromagnetic fields of the horizontal dipole in the presence of a spiral conductive unclosed sphere.

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The volumetric parametric resonance in magnetizable medium

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The stability of magnetizable medium stationary states at parametric excitation of a magnetic field is studied. Parameters of exited acoustic wave and the influence of oscillating magnetic field on the dispersion of sound and its propagation velocity are determined using asymptotic and numerical methods.
Keywords: parametric resonance, oscillating magnetic field.

Пацегон М. Ф., Поцелуєв С. І., **Об'ємний параметричний резонанс в середовищах, що намагнічуються.** Вивчається можливість втрати стійкості стаціонарних станів намагнічуваних середовищ при їх параметричному збудженні магнітним полем. Асимптотичними та чисельними методами встановлені параметри збуджуваних акустичних хвиль, вплив осцилюючої частини магнітного поля на дисперсність збуджуваного звуку та швидкість його поширення.
Ключові слова: параметричний резонанс, осцилююче магнітне поле.

Пацегон Н.Ф., Поцелуев С.И., **Объемный параметрический резонанс в намагничивающихся средах.** Изучается возможность потери устойчивости однородных состояний намагничивающихся сред при их параметрическом возбуждении магнитным полем. Асимптотическими и численными методами установлены параметры возбуждаемых акустических волн, влияние осциллирующей части магнитного поля на дисперсность возбуждаемого звука и скорость его распространения.
Ключевые слова: параметрический резонанс, осциллирующее магнитное поле.

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Introduction

Magnetic fluids (MFs) are widely used in modern acoustical devices in order to increase their capacity, selectivity of certain sound frequencies and to increase their operational resource [1]. There are possibilities to use ferrofluids as converters of acoustic oscillations [2], a study of the connection between acoustic properties of (MFs) and their structure are of the grate interest for physico-chemistry of disperse systems in order to obtain the information about the stability, reconstruction times of microstructure and irreversible phenomena in the process of structure formation [3].The known results of ferrofluid acoustics are reduced to the study of the influence of magnetic field on the propagation velocity and absorption of ultrasonic vibrations [4]. In this paper we investigate the possibility of new excitation mechanisms of acoustic vibrations in (MFs) during the loss of stability of homogeneous fluid stationary states in oscillating magnetic field. This paper continues the study, initiated in [5], and earlier studies of the stability of ferrofluid free surface in oscillating magnetic and gravitational fields [6, 7].

1. Basic equations

Magnetizable medium and electromagnetic field form closed thermodynamic system. Therefore, dynamic equations of magnetizable medium take the form of conservation laws [8]:

1. Mass conservation

$$\frac{\partial \rho}{\partial t} + div \rho \vec{v} = 0 \tag{1}$$

2. Momentum conservation

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial x_k}(\rho v_i v_k - p_{ik}) \tag{2}$$

3. Energy conservation

$$\frac{\partial}{\partial t}\left(\rho u + \rho \frac{v^2}{2}\right) = -div \vec{J}_e \tag{3}$$

4. Entropy balance equation

$$\frac{\partial}{\partial t}\rho\ s = -div \vec{J}_s + \sigma_s. \tag{4}$$

Here and below the following notation are introduced as: ρ is the density of medium, $\vec{v}$ is the velocity, $\{p_{ik}\}$ is the Cauchy symmetric stress tensor; u, s are the density of the internal energy and the entropy; $\vec{J}_e, \vec{J}_s$ are flux density vectors of the energy and the entropy, σ_s is internal entropy production, $div \equiv \vec{\nabla} \cdot ()$, $rot \equiv \vec{\nabla} \times ()$.

Equations (1)-(4) are supplemented by equations of quasi-stationary electrodynamics of non-conductive medium:

$$div\vec{B} = 0, \ \ rot\vec{H} = 0, \ \ \frac{\partial \vec{B}}{\partial t} = -c\, rot\vec{E}, \ \ \vec{B} = \vec{H} + 4\pi\vec{M}. \tag{5}$$

In equations (5) displacement currents are neglected, which is equivalently to the basic ferrohydrodynamics assumption about the same order of the characteristic frequency and size of changes of electromagnetic and hydrodynamic quantities.

Accepting the hypothesis of local equilibrium, the medium is concretized by the Gibbs identity in the form

$$du = Tds - pd\frac{1}{\rho} + \frac{\vec{H}}{4\pi}d\left(\frac{\vec{B}}{\rho}\right). \tag{6}$$

Here T is the temperature, p is the pressure, $\vec{H}, \vec{E}$ are strength of magnetic and electric fields, $\vec{B}$ is magnetic induction, $\vec{M}$ is the magnetization.

It should be noted that implementation of equation (6) does not depend on the way of magnetization of the medium (isotropic or anisotropic) [9].

Using methods of non-equilibrium thermodynamics [8], expressions for unknown flows in equations (1)-(4) are obtained

$$p_{ik} = -p\delta_{ik} + \tfrac{H_i B_k}{4\pi} + \tau_{ik};$$

$$J_{ek} = \rho v_k(u + \tfrac{p}{\rho} + \tfrac{v^2}{2}) + \tfrac{c}{4\pi}[\vec{E}^*, \vec{H}]_k - \tfrac{(\vec{v}\vec{H})B_k}{4\pi} - v_i\tau_{ik} + q_k; \tag{7}$$

$$\vec{J}_s = \rho s\vec{v} + \tfrac{\vec{q}}{T}; \quad \sigma_s = \tfrac{1}{T}(\tau_{ik}\tfrac{\partial v_i}{\partial x_k} - \vec{q}\nabla T).$$

Where $\{\tau_{ik}\}$ is the tensor of viscous stresses, $\vec{q}$ is the vector of heat flux density, $\vec{E}^* = \vec{E} + \frac{1}{c}[\vec{v}, \vec{B}]$ is the electric field strength in the proper reference frame.

Satisfying the second law of thermodynamics, i.e. inequality $\sigma_s \geq 0$, in the linear approximation of the Onsager theory, constitutive equations are obtained

$$\begin{gathered}\vec{q} = -\kappa\,\nabla T, \quad \tau_{ik} = 2\eta v_{ik} + (\varsigma - \tfrac{2}{3}\eta)v_{ee}\delta_{ik}\,, \\ \kappa \geq 0, \quad \eta \geq 0, \quad \varsigma \geq 0,\end{gathered} \tag{8}$$

where $\kappa\,, \eta\,, \varsigma$ are coefficients of conductivity, shear and bulk viscosities; $\{v_{ik}\}$ is the strain rate tensor.

Equations (1)-(5) should be supplemented by equations of the thermodynamic state. To obtain them, the thermodynamic potential f is introduced as

$$f = u - Ts - \frac{\vec{B}\vec{H}}{4\pi\rho}. \tag{9}$$

From the Gibbs equation (6) follows

$$df = -sdT + \frac{p}{\rho^2}d\rho - \frac{\vec{B}d\vec{H}}{4\pi\rho}.$$

Therefore

$$s = -(\frac{\partial f}{\partial T})_{\rho,\vec{H}}; \quad p = \rho^2(\frac{\partial f}{\partial \rho})_{T,\vec{H}}; \quad \vec{B} = 4\pi\rho(\frac{\partial f}{\partial \vec{H}})_{\rho,T},$$

thus

$$s = s(\rho, T, \vec{H}); \quad p = p(\rho, T, \vec{H}); \quad \vec{B} = \vec{B}(\rho, T, \vec{H})$$

is the most common form of equations of the thermodynamic state. The final equation determines the magnetization law of the medium.

For isotropic magnetizable medium equations of state have following form:

$$\begin{aligned}
&\vec{B} = \mu\vec{H}; \mu = \mu(\rho, T, H);\\
&f = f^0(\rho, T) - \tfrac{1}{4\pi}\int\limits_0^H \mu(\rho, T, H)HdH;\\
&p = p^0(\rho, T) + \psi\ ; \ \ s = s^0(\rho, T) + s^{(m)}\\
&\psi \ = \ \tfrac{1}{4\pi}\int\limits_0^H [\mu - \rho(\tfrac{\partial\mu}{\partial\rho})_{T,H}]HdH\ ;\\
&s^{(m)} \ = \ \tfrac{1}{4\pi\rho}\int\limits_0^H (\tfrac{\partial\mu}{\partial T})_{\rho,H}HdH;\\
&u = u^0(\rho, T) + \tfrac{BH}{4\pi\rho} - \tfrac{1}{4\pi}\int\limits_0^H (\mu - T\mu_T)HdH.
\end{aligned} \tag{10}$$

Here μ is a magnetic permeability of the medium; the expression $f_\psi := \partial f/\partial\psi$ denotes the corresponding partial derivative, index " 0 " at the top marked thermodynamic functions of the medium in the absence of the field. These functions, which assumed known, satisfy the Gibbs equation in the absence of the field

$$du^0 = Tds^0 - p^0 d\frac{1}{\rho}.$$

Equations (1)-(5), (7)-(10) form a closed system of equations of the medium dynamics with the equilibrium magnetization and written as [11],[15]:

$$\begin{gathered}
\tfrac{d\rho}{dt} + \rho\, div\vec{v} = 0,\\
\rho\tfrac{d\,\vec{v}}{dt} = -\nabla p + M\nabla H + \eta\Delta\vec{v} + \left(\varsigma + \tfrac{1}{3}\eta\right)\nabla div\,\vec{v},\\
\rho T\tfrac{ds}{dt} = \kappa\Delta T + 2\eta v_{ik}v_{ik}\,,\\
div\vec{B} = 0, \quad rot\vec{H} = 0\,, \quad \vec{B} = \mu\vec{H}\,, \quad \mu = \mu(\rho, T, H)\,,\\
p = p^0(\rho, T) + \psi\,, \quad s = s^0(\rho, T) + s^{(m)}\,.
\end{gathered} \tag{11}$$

By virtue of (6), instead of the entropy equation in this system can be used the energy equation in the form

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\left(\rho u + \rho\tfrac{v^2}{2}\right) = -\tfrac{\partial}{\partial x_k}\Big[\rho v_k\Big(u + \tfrac{p}{\rho} + \tfrac{v^2}{2} -\\
&-\tfrac{(\vec{B}\vec{H})}{4\pi\rho}\Big) + \tfrac{c}{4\pi}\left[\vec{E}\times\vec{H}\right]_k - \kappa\,\tfrac{\partial T}{\partial x_k} - v_i\tau_{ik}\Big]\,.
\end{aligned} \tag{12}$$

2. Effective nonmagnetic medium, corresponding to magnetizable medium

One-dimensional unsteady motion of a magnetizable medium along x axis is considered. Then $v_x = v, v_y \equiv 0, v_z \equiv 0$ and besides

$$v = v(x,t), \quad \rho = \rho(x,t), \quad T = T(x,t), \quad \vec{H} = \vec{H}(x,t).$$

From equations of electrodynamics (5) follows

$$B_x = B_x(t), \quad H_y = H_y(t), \quad H_z = H_z(t).$$

Denote

$$B_x(t) = \chi_1(t), \; H_y(t) = \chi_2(t), \; H_z(t) = \chi_3(t).$$

Functions $\chi_i(t)$ are determined by boundary conditions.

Equations of motion (2) are reduced to the form:

$$\begin{gathered} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho v = 0, \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x} + \left(\zeta + \frac{4}{3}\eta\right)\frac{\partial^2 v}{\partial x^2}, \\ \rho T\frac{ds}{dt} = \kappa\frac{\partial^2 T}{\partial x^2} + 2\eta\left(\frac{\partial v}{\partial x}\right)^2. \end{gathered} \tag{13}$$

Taking into account that

$$div(\vec{E}\times\vec{H}) = \vec{H}rot\vec{E} - \vec{E}rotH = -\frac{1}{c}\vec{H}\frac{\partial \vec{B}}{\partial t}$$

the energy equation (12) is written as

$$\frac{\partial}{\partial t}\left(\rho u_e + \rho\frac{v^2}{2}\right) = -\frac{\partial}{\partial x}\left[\rho v(u_e + \frac{p_e}{\rho} + \frac{v^2}{2}) - \kappa\frac{\partial T}{\partial x} - v\tau_{11}\right] + \rho q.$$

Here the following notations are introduced:

$$\begin{gathered} p_e = p - \frac{\chi_1^2}{4\pi\mu}; \\ u_e = u - \frac{\mu}{4\pi\rho}(\chi_2^2 + \chi_3^2); \\ \rho q = \frac{1}{8\pi\mu}\frac{d}{dt}\chi_1^2 - \frac{\mu}{8\pi}\frac{d}{dt}(\chi_2^2 + \chi_3^2). \end{gathered} \tag{14}$$

Thus, equations of one-dimensional motion of magnetizable medium are reduced to equations of one-dimensional gas dynamics with special equations of state.

Equations have this form regardless from the way of magnetization (isotropic or anisotropic). It affects only on the equation of state, i.e. function p_e, u_e. The energy equation (14) differs from the ordinary equation of gas dynamics by the presence of term ρq on the right-hand side. Note that $q = 0$ if $\chi_i = const$ and this case was considered in [10]. The value $q \neq 0$ can be interpreted as a mass density of energy sources in the medium. This especially becomes clear from the Gibbs equation (6), which for one-dimensional motions of magnetizable media can be written as

$$du_e = Tds - p_e d\frac{1}{\rho} + qdt. \tag{15}$$

In the equation (15) the magnetic field strength is excluded. If $q = 0\,(\chi_i = const)$ this corresponds to a two-parametric medium with constitutive parameters: ρ and s, mass density of internal energy u_e and pressure p_e, besides

$$u_e = u_e(\rho, s); \quad p_e = p_e(\rho, s).$$

At $q \neq 0$ functions $\chi_i = \chi_i(t)$ are given by appropriate boundary conditions. They determine the energy exchange between the nonmagnetic medium and external bodies. They can be considered as external control of nonmagnetic medium from the external system, which is the magnetic field.

Nonmagnetic medium, defined by equations of state (14), below will be called an effective medium, corresponding to the initial magnetizable medium.

Equations (14) can be written in the form:

$$p_e(\rho, s, t) = p^0(\rho, T) - \frac{\chi_1^2}{4\pi\mu} + \frac{1}{4\pi}\int_0^H (\mu - \rho\mu_\rho)HdH,$$

$$u_e(\rho, s, t) = u^0(\rho, T) + \frac{\chi_1^2}{4\pi\rho\mu} - \frac{1}{4\pi\rho}\int_0^H (\mu - T\mu_T)HdH.$$

The temperature and the magnetic field strength in the right-hand side of equations must be excluded using relations:

$$T = T(\rho, s, \chi_i); \quad H = H(\rho, s, \chi_i)\,;$$

$$\chi_i := B_x(t)\,,\ H_y\,(t)\,,\ H_z\,(t).$$

To obtain them it is necessary to solve for T , H the following system of nonlinear functional equations:

$$\Phi = \mu(\rho, T, H)H - [\chi_1^2 + \mu^2(\rho, T, H)(\chi_2^2 + \chi_3^2)]^{\frac{1}{2}} = 0,$$

$$\Psi = s - s^0(\rho, T) - \frac{1}{4\pi}\int_0^H \mu_T HdH = 0.$$

Conditions for the solvability of this system of equation for the T, H consist of the inequality

$$\frac{\partial(\Phi, \Psi)}{\partial(T, H)} \neq 0,$$

which assumed to be satisfied.

Thus, in the case of a linear isotropic magnetization, taking into account the magnetocaloric effect ($\mu = \mu(\rho, T)$), we have:

$$s = s^0(\rho, T) + \frac{1}{8\pi\mu^2\rho}[\mu^2(\chi_2^2 + \chi_3^2) + \chi_1^2]\mu_T,$$
$$H = [\chi_2^2 + \chi_1^2 + \mu^{-2}\chi_1^2]^{\frac{1}{2}}.$$

Then from the first equation the dependence $T = T(\rho, s, \chi_i)$ can be determined and the second equation gives necessary relation $H = H(\rho, s, \chi_i)$.

After that, equations of state of an effective medium are determined:

$$p_e = p_e(\rho, s, \chi_i) = p^0(\rho, T) + \frac{1}{8\pi\mu^2}[\mu^2(\mu - \\ -\rho\mu_\rho)(\chi_2^2 + \chi_3^2) - (\mu - \rho\mu_\rho)\chi_1^2],$$
$$u_e = u_e(\rho, s, \chi_i) = u^0(\rho, T) - \frac{1}{8\pi\mu^2}[\mu^2(\mu - \\ -T\mu_\rho)(\chi_2^2 + \chi_3^2) - (\mu - T\mu_\rho)\chi_1^2].$$

If a non-linear law of magnetization is considered and magnetocaloric effect can be neglected, i.e. $\mu = \mu(\rho, H)$, then $s = s^0(\rho, T)$, $T = T(\rho, s)$ and the dependence $H(\rho, \chi_i)$ is directly determined by the law of magnetization.

As equations of state of the effective medium depend on the time explicitly, such medium is non-stationary. This kind of medium has recently been studied in electrodynamics [12]. It should be noted that equations (13)-(15) are essentially nonlinear even in the case of an ideal medium because $p_e = p_e(\rho, s, t)$. They are quasi-linear only in the case $\chi_i = const$.

3. Excitation of acoustic vibrations in oscillating magnetic field

At non-stationary parameters $\chi_i = \chi_i(t)$ the equation (13) allows stationary homogeneous solution:

$$\rho \equiv \rho_0, \quad v = v_0 \equiv 0, \quad s \equiv s_0 = const.$$

In this case, the energy enters to effective medium according to the equation

$$\frac{\partial u_e}{\partial t} = q(t).$$

If magnetocaloric effect is neglected, the temperature of the medium will be constant: $T = T_0$. But when this effect is taken into account the temperature of the homogeneous state depends on the time: $T = T(t)$, so that the condition of adiabaticity is performed ($s = s_0 = const$). Furthermore, the magnetic field is homogeneous: $\vec{H} = \vec{H}(t)$. Depending on the type of source $q(t)$ in the magnetizable medium new effects, that have not previously been studied, become possible.

The solution of system (13) for not heat-conducting medium ($\kappa = 0$) is sought in the form

$$\rho = \rho_0 + \rho'(x,t), \quad v = v'(x,t),$$

where the prime denotes the perturbation of parameters.

By linearizing of equations (13) relative to homogeneous state, we obtain:

$$\begin{gathered} \frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v'}{\partial x} = 0\,; \\ \rho_0 \frac{\partial v'}{\partial t} + a^2 \frac{\partial \rho'}{\partial x} + \left(\zeta + \frac{4}{3}\eta\right) \frac{\partial^2 v'}{\partial x^2} = 0; \\ s \equiv s_0 = const. \end{gathered} \tag{16}$$

Here $a^2 = (\frac{\partial p_e}{\partial \rho})_{s,\chi_i(t)}$, i.e. derivative of the effective pressure is calculated at constant entropy s and given functions $\chi_i(t)$. Therefore

$$a^2 = a^2(\rho_0, s_0, \chi_1(t), \chi_2(t), \chi_3(t)) = a^2(t).$$

As shown in [10], a^2 is equal to the square of the velocity of sound propagation in magnetizable medium and given by the following expression [4]:

$$\begin{aligned} & a^2(\rho, t) = L_0 - L_1 (1 + L_2)^{-1}\,; \\ & L_0 = \rho x_{31} + x_{23} x_{32}\,; \\ & L_1 = 4\pi\rho\mu^3 m^2 \left[\chi_1^2(t) + \mu^{-2}(\chi_2^2(t) + \chi_3^2(t))\right] \times \\ & \times(\rho\mu_\rho + N\mu_T x_{23}) \left[\rho(\mu_\rho + \mu_T T_\rho) + \mu_T T_{s^0} x_{23}\right]; \\ & L_2 = (\mu^2 \mu_T^2 T_{s^0} N m \left[\chi_1^2(t) + \mu^{-2}(\chi_2^2(t) + \chi_3^2(t))\right] - \\ & - \mu^2 \mu_H B^{-1}(\chi_2^2(t) + \chi_3^2(t)))(\mu^2 + \mu_H B)^{-1}; \\ & m^{-1} = 4\pi\rho\mu(\mu^2 + \mu_H B)\,; \\ & N^{-1} = 1 + T_{s^0}(s_T^m - \mu_T m B^2)\,; \\ & x_{23} = \rho N \left[m\mu_T B^2(\mu_\rho + \mu_T T_\rho) - s_\rho^m - s_T^m T_\rho\right]; \\ & x_{31} = (p_\rho^0 + \psi_\rho + \psi_T T_\rho)/\rho + \rho\mu_\rho m B^2(\mu_\rho + \mu_T T_\rho)\,; \\ & x_{32} = (p_{s^0}^0 + \psi_T T_{s^0})/\rho + \rho T_{s^0}\mu_\rho \mu_T m B^2\,; \\ & T_{s^0} = \left(\frac{\partial T}{\partial s^0}\right)_\rho;\; p_{s^0}^0 = \left(\frac{\partial p^0}{\partial s^0}\right)_\rho. \end{aligned} \tag{17}$$

Due to the potentiality of one-dimensional motions $v' = \partial\varphi/\partial x$, where $\varphi = \varphi(x,t)$ is the velocity potential. Then from the second equation of (16) the equation for density perturbations is obtained

$$\rho' = -\frac{\rho_0}{a^2}\left(\frac{\partial \varphi}{\partial t} + \nu_0 \frac{\partial^2 \varphi}{\partial x^2}\right), \quad \nu_0 = \frac{1}{\rho_0}\left(\zeta + \frac{4}{3}\eta\right). \tag{18}$$

This allows to get from the first equation of (16) the following equation for velocity potential

$$\frac{\partial^2 \varphi}{\partial t^2} - a^2 \frac{\partial^2 \varphi}{\partial x^2} + \nu_0 \frac{\partial^3 \varphi}{\partial x^2 \partial t} - \left[\frac{\partial \varphi}{\partial t} + \nu_0 \frac{\partial^2 \varphi}{\partial x^2}\right] \frac{d}{dt}(\ln a^2) = 0. \tag{19}$$

Trivial solution $\varphi = const$ of this equation corresponds to the equilibrium state of magnetic fluid $\rho = const; v = 0$. The stability analysis of this equilibrium state is performed below.

The solution of equation (19) is sought in the form

$$\varphi(x,t) = \varphi(t)e^{ikx}.$$

For the amplitude of the perturbation $\varphi(t)$ we get

$$\ddot{\varphi} + \left[k^2\nu_0 - \frac{d}{dt}\ln(a^2)\right]\dot{\varphi} + k^2\left[a^2 - \nu_0\frac{d}{dt}\ln(a^2)\right]\varphi = 0. \tag{20}$$

For further study of equation (20) it is necessary to specify the explicit form of $a^2(t)$, given by the expression (17). In the case of general isotropic law of magnetization, equations for equilibrium state of effective medium can be obtained only by using numerical methods. For the study of qualitative characteristics of excited acoustic oscillations in magnetic fluids, the most important case of isotropic magnetization is considered.

For an ideal paramagnet the magnetization is determined by the Langevin equation [11]:

$$M = mnL(\xi), \quad \xi = \frac{mH}{kT}, \quad L = cth\xi - \xi^{-1},$$

where: m is the magnetic momentum of ferromagnetic particle, n is the volume concentration, k is the Boltzmann constant.

Then in weak fields ($\xi << 1$) we obtain

$$\mu = 1 + \alpha\rho;\ \alpha = \frac{4\pi c_1 m^2}{3\mathcal{M}kT},$$

where c_1 is mass concentration of magnetic particles, $\mathcal{M}$ is the mass of a single ferromagnetic particle. If the temperature changes are neglected: $\alpha = const$. Then

$$a^2 = a_0^2 + \frac{(\mu-1)^2}{8\pi\mu^3}\chi_1^2; \tag{21}$$

$$p_e = p^0(\rho, s^0) + \tfrac{1}{8\pi}(\chi_2^2 + \chi_3^2) - \tfrac{2\mu-1}{8\pi\mu^2}\chi_1^2; \quad s^0 = s_0 = s; \quad a_0^2 = \tfrac{\partial p^0(\rho,s^0)}{\partial\rho}.$$

Here a_0^2 is the square of sound velocity in the medium in the absence of a magnetic field.

In this case it is obtained, that the magnetic field components, perpendicular to the direction of wave propagation, do not affect on the velocity of sound propagation. Moreover, the velocity of sound propagation along magnetic field direction is greater than in the absence of the field.

Suppose that the parameter χ_{10} is time-dependent according to harmonic law

$$\chi_1 = \chi_{10} + \beta\cos 2\omega t. \tag{22}$$

Then for the sound velocity in the medium we have

$$a^2(t) = a_0^2 + \frac{(\mu-1)^2}{4\pi\rho\mu^3}\left(\chi_{10}^2 + \frac{\beta^2}{2} + 2\chi_{10}\beta\cos 2\omega t + \frac{\beta^2}{2}\cos 4\omega t\right), \qquad (23)$$

where $a_0 = a_0(\rho_0, s_0)$, $\mu = \mu(\rho_0)$ are constant parameters, determined at equilibrium state. By substituting (23) in (20), the following equation is obtained

$$\begin{aligned} &\tfrac{d^2\varphi}{d\tau^2} + [\psi_0 + 2\psi_{2s}\sin 2\tau + 2\psi_{4s}\sin 4\tau]\tfrac{d\varphi}{d\tau} + \\ &+[\theta_0 + 2\theta_{2s}\sin 2\tau + 2\theta_{4s}\sin 4\tau + 2\theta_{2c}\cos 2\tau + 2\theta_{4c}\cos 4\tau]\,\varphi = 0, \end{aligned} \qquad (24)$$

where

$$\psi_0 = \tfrac{k^2\nu_0}{\omega},\ \ \psi_{2s} = \tfrac{(\mu-1)^2\chi_{10}}{2\pi\rho\mu^3A^2}\beta,\ \ \psi_{4s} = \tfrac{(\mu-1)^2}{4\pi\rho\mu^3A^2}\beta^2,$$

$$\theta_0 = \tfrac{k^2}{\omega^2}A^2,\ \ \theta_{2s} = \tfrac{(\mu-1)^2k^2\nu_0\chi_{10}}{2\pi\rho\mu^3\omega A^2}\beta,\ \ \theta_{2c} = \tfrac{(\mu-1)^2k^2\chi_{10}}{4\pi\rho\mu^3\omega^2}\beta,$$

$$\theta_{4s} = \tfrac{(\mu-1)^2k^2\nu_0}{4\pi\rho\mu^3\omega A^2}\beta^2,\ \ \theta_{4c} = \tfrac{(\mu-1)^2k^2}{16\pi\rho\mu^3\omega^2}\beta^2,\ \ A^2 = a_0^2 + \tfrac{(\mu-1)^2}{4\pi\rho\mu^3}\chi_{10}^2,$$

$\tau = \omega t$ is dimensionless time.

The equation (24) has periodic solutions, corresponding to acoustic waves.

4. Asymptotic solution

The equation (20) by substitution

$$\varphi(\tau) = a\,Z\exp\left(-\frac{k^2\nu_0\tau}{2\omega}\right) \qquad (25)$$

is reduced to the form

$$\frac{d^2Z}{d\tau^2} + \left[\frac{k^2a^2}{\omega^2} - \frac{2\nu_0k^2}{\omega a}\frac{da}{d\tau} - \left(\frac{k^2\nu_0}{16\omega} - \frac{1}{8a}\frac{da}{d\tau}\right)^2 + \frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right)\right]Z = 0.$$

In the case of time-dependent sound velocity in the form (23), by neglecting of terms of order β^2, the Hill equation for the function Z is obtained

$$\frac{d^2Z}{d\tau^2} + [\theta_0 - \psi_0^2 + 2(\theta_{1c} - \psi_{1s})\cos 2\tau + \theta_{1s}\sin 2\tau)]\,Z = 0. \qquad (26)$$

In the first approximation by the small parameter β marginal stability curves of the first unstable region is given by

$$\theta_0 = 1 + \psi_0^2 \pm ((\theta_{1c} - \psi_{1s})^2 + \theta_{1s}^2)^{1/2}.$$

As follows from (25), it is necessary to find an unstable solution of the equation (26). Using the method of Whittaker [13, 14], as a first approximation is taken

$$Z \;=\; e^{\gamma\tau}\,\sin(\tau-\sigma). \tag{27}$$

By substituting (27) in (26) and equating coefficients at $\sin\tau$ and $\cos\tau$, for the first unstable region is obtained

$$\begin{aligned} 2\gamma \;&= (\theta_{1c}-\psi_{1s})\sin 2\sigma - \frac{\theta_{1s}}{2}\cos 2\sigma,\\ \theta_0 &= 1+\psi_0^2-\gamma^2+\,(\theta_{1c}-\psi_{1s})\cos 2\sigma\;+\;\frac{\theta_{1s}}{2}\sin 2\sigma. \end{aligned} \tag{28}$$

From this

$$\begin{aligned} \gamma^2 \;&=\; -(1+\theta_0-\psi_0^2)\pm\left(4(\theta_0-\psi_0^2)+(\theta_{1c}-\psi_{1s})^2+\tfrac{\theta_{1s}^2}{4}\right)^{1/2},\\ tg\sigma \;&=\; \frac{(\theta_{1c}-\psi_{1s})\pm[(\theta_{1c}-\psi_{1s})^2+\frac{\theta_{1s}^2}{4}-4\gamma^2]^{1/2}}{2\gamma-\theta_{1s}/2}. \end{aligned} \tag{29}$$

Values $\gamma^2 \geq 0$, $0 \leq \sigma \leq \pi/2$ correspond to unstable solutions.

In the first approximation, in view of (25) and (27), the solution of the equation (20) is obtained

$$\varphi(t) \;=\; A\,\exp((\gamma\omega - k^2\nu_0/2)t)\,\sin(\omega t-\sigma).$$

This is periodic solution if the following condition

$$\gamma \;=\; \frac{k^2\nu_0}{2\omega}$$

is satisfied. Then the equation (19) for the velocity potential has periodic solution

$$\varphi(x,t) \;=\; A\,\exp(i(kx-\omega\,t+\sigma))\,,$$

which corresponds to the potential of small-amplitude waves, excited as a result of parametric instability, and propagating at the velocity ω/k. The frequency of excited waves is twice less then frequency of the parametric excitation.

Taking into account (29), the equation, that determines the magnitude of the wave vector depending on parametric excitation frequency, is obtained:

$$\begin{aligned} &\left(1-\tfrac{k^2A^2}{\omega^2}\right)^2+\tfrac{1}{16}\left(\tfrac{k^2\nu_0}{\omega}\right)^2\left[9\left(\tfrac{k^2\nu_0}{\omega}\right)^2+40\right]-\tfrac{3}{2}\tfrac{k^2A^2}{\omega^2}\,\tfrac{k^2\nu_0}{\omega}\;=\\ &=\left[\tfrac{\beta\chi_0(\mu-1)^2}{4\pi\rho\mu^3A^2}\right]^2\left[\left(\tfrac{k^2A^2}{\omega^2}-2\right)^2+\left(\tfrac{k^2\nu_0}{\omega}\right)^2\right]. \end{aligned} \tag{30}$$

Hence it follows that the excited waves are dispersive and the dispersion is a result of the viscosity of the medium.

In the case of an ideal medium the equation (30) has two solutions

$$\frac{\omega^2}{k^2} = A^2\,(1 \pm \varepsilon), \quad \varepsilon = \frac{\beta\chi_0(\mu-1)^2}{4\pi\rho\mu^3 A^2}. \tag{31}$$

Thus in this case waves propagate without dispersion. As A is the wave velocity in a constant field, the oscillating part of the magnetic field can leads to either increase or decrease of their velocity.

Values (31) correspond to periodic solutions of the equation (26) and the value of parameters, which belong to the boundaries of stability regions. Therefore, at the same frequency of the magnetic field can be excited waves of different lengths.

5. Numerical solution

For the case of weak magnetic fields ($\xi << 1$) the equation (24) for the velocity potential was obtained. The equation (24) includes periodic functions of time, so the solution of this equation is sought in the Floquet form

$$\varphi(\tau) = e^{\gamma\tau} Y(\tau),$$

where $\gamma = s + i\alpha$ is the Floquet exponent; $Y(\tau)$ is a periodic function with period $\frac{\pi}{\omega}$, therefore it can be expanded in the Fourier series

$$Z(\tau) = \sum_{n=-\infty}^{\infty} \phi_{2n} e^{2n\tau i}.$$

Then

$$\varphi(\tau) = \sum_{n=-\infty}^{\infty} \phi_{2n} e^{q_{2n}\tau}, \quad q_{2n} = s + i(\alpha + 2n). \tag{32}$$

By substituting (32) in (24), we obtain

$$\begin{gathered}\sum_{n=-\infty}^{\infty} e^{q_{2n}\tau}[(q_{2n}^2 + q_{2n}\psi_0 + \theta_0)\phi_{2n} + \\ +(\theta_{2c} - i(\theta_{2s} + q_{2n}\psi_{2s}))\phi_{2n+1} + (\theta_{2c} + i(\theta_{2s} + q_{2n}\psi_{2s}))\phi_{2n-1} + \\ +(\theta_{4c} - i(\theta_{4s} + q_{2n}\psi_{4s}))\phi_{2n+2} + (\theta_{4c} + i(\theta_{4s} + q_{2n}\psi_{4s}))\phi_{2n-2}] = 0.\end{gathered} \tag{33}$$

In matrix form (33) can be written as

$$(C + \beta B + \beta^2 D)\phi = 0, \tag{34}$$

where C is diagonal matrix with complex coefficients, B and D are banded matrices with two and three subdiagonals:

$$C = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ & & & & \iddots \\ \dots & c_{-1,-1} & 0 & 0 & \dots \\ \dots & 0 & c_{0,0} & 0 & \dots \\ \dots & 0 & 0 & c_{1,1} & \dots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}; B = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & 0 & b_{-1,0} & 0 & 0 & \dots \\ \dots & b_{0,-1} & 0 & b_{0,1} & 0 & \dots \\ \dots & 0 & b_{1,0} & 0 & b_{1,2} & \dots \\ \dots & 0 & 0 & b_{2,1} & 0 & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix};$$

$$D = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & d_{-1,-1} & 0 & d_{-1,1} & 0 & \dots \\ \dots & 0 & d_{0,0} & 0 & d_{0,2} & \dots \\ \dots & d_{1,-1} & 0 & d_{1,1} & 0 & \dots \\ \dots & 0 & d_{2,0} & 0 & d_{2,2} & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix};$$

$$c_{n,n} = q_{2n}^2 + \frac{k^2\nu_0}{\omega} q_{2n} + \frac{k^2}{\omega^2}\left(a_0^2 + \frac{(\mu-1)^2}{4\pi\rho\mu^3}\chi_{10}^2\right); d_{n,n} = \frac{(\mu-1)^2}{8\pi\rho\mu^3}\frac{k^2}{\omega^2};$$

$$b_{n,n\pm1} = \frac{(\mu-1)^2}{2\pi\rho\mu^3}\left[\frac{k^2}{2\omega^2} \mp \frac{i(q_{2n} + \frac{\nu_0 k^2}{\omega})}{a_0^2 + (\mu-1)^2\chi_0^2/(4\pi\rho\mu^3)}\right]\chi_{10};$$

$$d_{n,n\pm2} = \frac{(\mu-1)^2}{4\pi\rho\mu^3}\left[\frac{k^2}{4\omega^2} \mp \frac{i(q_{2n} + \frac{\nu_0 k^2}{\omega})}{a_0^2 + (\mu-1)^2\chi_0^2/(4\pi\rho\mu^3)}\right].$$

In the case of pure oscillating magnetic field $\chi_{10} = 0 : b_{n,n\pm1} = 0$. Then by inverting of the matrix C, from (34) follows the ordinary eigenvalue problem:

$$(C^{-1}D)\phi = \frac{1}{\beta^2}\phi. \tag{35}$$

At the stability analysis is usually used the following procedure [16]: the first step is to fix the wavenumber k and the amplitude β, as well as values of other hydrodynamic parameters of the system, and then the Floquet exponent $\gamma = s + i\alpha$ is calculated. Marginal stability curves in the plane (k, β) are curves on which $s(k, \beta) = 0$. This condition is satisfied by interpolation of β at fixed k between negative and positive values of s.

But in our calculations the method described in [17] is used: the Floquet exponent $\gamma = s + i\alpha$ is pre-fixed, then the eigenvalue problem (35) is solved at fixed value of k. The largest real positive eigenvalue $\frac{1}{\beta^2}$, corresponding to a minimum amplitude β, is sought by interpolation of k. To construct marginal stability curves in the plane (k, β) we have to set $s = 0$ and $\alpha = 0$ $(\alpha = 1)$, which corresponds to the case of harmonic (subharmonic) oscillations. The above method for calculation of boundaries of instability regions is used to solve the problem (35). Matrices A and D are cut to size, providing the required accuracy of calculations. In all calculations the typical ferrofluid parameters were accepted

$$\nu = 0.1(\mathrm{P}),\ \mu = 2,\ \sigma = 30\left(\frac{\mathrm{erg}}{\mathrm{cm}^2}\right),\ \rho = 1.2\left(\frac{\mathrm{g}}{\mathrm{cm}^3}\right),\ a_0 = 1.5\cdot10^5\left(\frac{\mathrm{cm}}{\mathrm{s}}\right).$$

Boundaries of the first two unstable regions (the Ince-Strutt diagram for a viscous fluid) is shown on Fig.1.a) and Fig.1.b). Marginal stability curves form narrow regions ("tongues"), the value of parameters outside (inside) of these regions corresponds to stability (instability). The absolute minimum of this curves

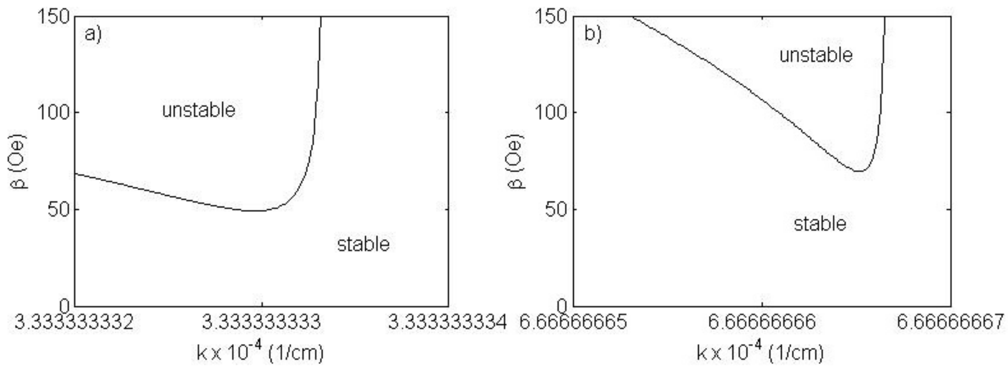

Fig. 1. a) The first and b) the second region of parametric instability at excitation frequency $\omega = 100$ (Hz) of magnetic field.

determines the critical wavenumber k_c and the critical amplitude β_c, at which instability occurs.

Fig.2.a) shows, that if magnetic field frequency increases, acoustic waves with less wavelength are excited. Moreover, at increasing of frequency for excitation of parametric instability must be applied the oscillating field of greater amplitude (see Fig.2.b))

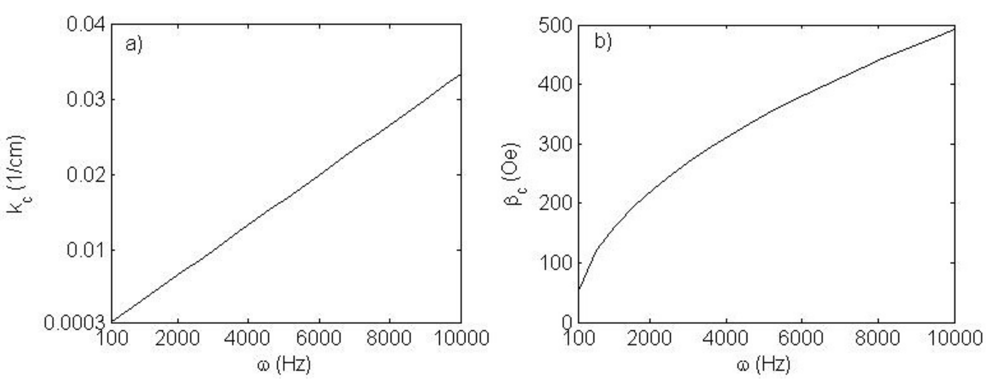

Fig. 2. The dependence of a) the critical wavenumber k_c and b) the critical amplitude β_c on the frequency ω of oscillating magnetic field.

In the case, when the magnetic field consist of constant and oscillating parts, the eigenvalue problem (34) must be solved. Using the column vector $\phi := \beta\xi$, the equation (34) reduces to the ordinary eigenvalue problem for matrix doubled in size

$$\begin{pmatrix} -D^{-1}B & -D^{-1}C \\ I & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \xi \end{pmatrix} = \beta \begin{pmatrix} \phi \\ \xi \end{pmatrix}, \tag{36}$$

where I is the identity matrix, which has the same size as B, C and D.

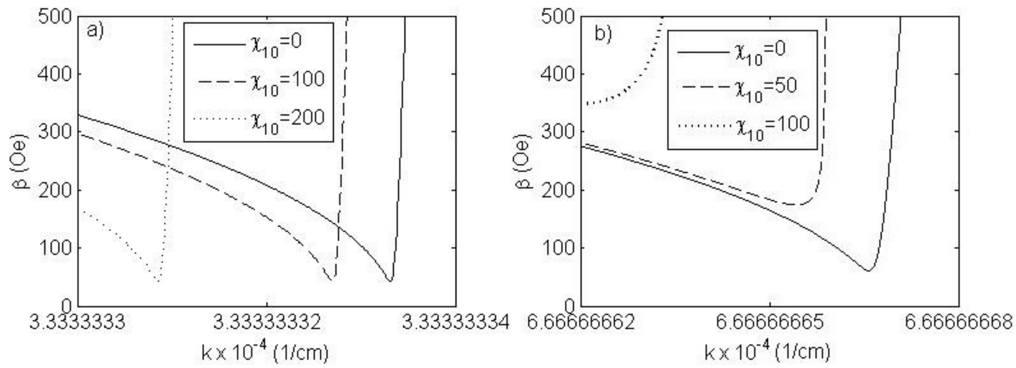

Fig. 3. a) The first and b) the second region of parametric instability for different values of stationary field χ_{10} and frequency $\omega = 100$ (Hz).

Similarly to the previous case, to construct regions of parametric instability in the plane of parameters (k, β) at fixed values of χ_{10}, the smallest real positive eigenvalue β of the problem (36) is sought. The calculation revealed that stationary component of the magnetic field has less (greater) impact on the structure of odd (even) instability regions. For the first unstable region at increasing of χ_{10} critical amplitude β_c remains almost unchanged, but exited sound waves have larger wavelength (see Fig.3.a)). Whereas for the second unstable region Fig.3.b) shows, that if χ_{10} increases, the critical amplitude β_c also increases, i.e. instability threshold shifts to higher values.

Conclusions

The parametric instability of ferrofluid volumes in weak homogeneous magnetic field, which consist of constant and oscillating parts, is considered. The appearance of unstable zones is studied. The problem was reduced to the Hill equation, which is studied using asymptotic and numerical methods. Marginal stability curves, that form narrow unstable regions corresponding to acoustical oscillations in ferrofluid, were obtained. The dependence of a structure of unstable tongues on the frequency ω and constant part χ_{10} of magnetic field is studied. It is shown, that increasing of ω leads to increasing of critical wavenumber k_c and critical amplitude β_c of magnetic field, required for the onset of instability. Also the increasing of χ_{10} causes to the appearance of shorter wavelength and can shifts a threshold of instability.

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Global synthesis of bounded controls for systems with power nonlinearity

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In this work we consider the problem of global bounded control synthesis for a nonlinear system with uncontrollable first approximation. A class of bounded controls that steer the system from any initial state to the origin in some finite time is constructed based on the controllability function method.
Key words: synthesis problem, finite-time stabilization, nonlinear systems, controllability function method.

Бебія М. О., **Глобальний синтез обмежених керувань для систем зі степеневою нелінійністю.** У роботі розглядається задача глобального синтезу обмежених керувань для нелінійної некерованої за першим наближенням системи. На основі методу функції керованості побудовано клас обмежених керувань, які переводять систему із довільного начального стану у початок координат за скінченний час.
Ключові слова: задача синтезу, стабілізація за скінченний час, нелінійні системи, метод функції керованості.

Бебия М. О., **Глобальный синтез ограниченных управлений для систем со степенной нелинейностью.** В работе рассматривается задача глобального синтеза ограниченных управлений для нелинейной неуправляемой по первому приближению системы. На основе метода функции управляемости построен класс ограниченных управлений, которые переводят систему из произвольного начального состояния в ноль за конечное время.
Ключевые слова: задача синтеза, стабилизация за конечное время, нелинейные системы, метод функции управляемости.

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1. Introduction

The problem of control design for nonlinear systems has been paid much attention in recent years [1]—[12]. In the present paper we consider a class of nonlinear systems with uncontrollable first approximation. Such systems play important role in control theory since most actual dynamical systems are inherently nonlinear.

Consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = u, \quad |u| \le d, \\ \dot{x}_i = c_{i-1}x_{i-1}, \quad i = 2, \dots, n-1, \\ \dot{x}_n = c_{n-1}x_{n-1}^{2k+1}, \end{cases} \tag{1}$$

where $k = \frac{p}{q}$, $p > 0$ is an integer, $q > 0$ is an odd integer, $u \in \mathbb{R}$ is a control, c_i, $i = 1, \dots, n-1$ are real numbers such that $\prod_{i=1}^{n-1} c_i \neq 0$, $d > 0$ is a given number.

System (1) is not stabilizable with respect to the first approximation. The stabilization problem for system (1) with $c_i = 1$, $i = 1, \dots, n-1$, and $k \in \mathbb{N}$ was solved in [4]. In the present paper we consider the problem of global synthesis of bounded controls for system (1). For the sake of brevity this problem will be referred as the global synthesis problem.

The global synthesis problem for system (1) is to find a control $u = u(x)$ such that

(i) for every $x_0 \in \mathbb{R}^n$ there exists a number $T(x_0) < +\infty$ such that $\lim\limits_{t \to T(x_0)} x(t, x_0) = 0$, where $x(t, x_0)$ is a solution of system (1) with $u = u(x)$ that satisfies the condition $x(0, x_0) = x_0$;

(ii) the control $u(x)$ satisfies the restriction $|u(x)| \le d$ for all $x \in \mathbb{R}^n$.

The control law construction is based on the controllability function method, which was proposed by V.I. Korobov [2] for a nonlinear system of the form

$$\dot{x} = \varphi(t, x, u), \quad x \in \mathbb{R}^n, \quad u \in \Omega \subset \mathbb{R}^r, \quad 0 \in int\, \Omega, \tag{2}$$

where $\varphi(t, 0, 0) = 0$ for all $t \ge 0$.

Consider the case $\frac{\partial \varphi(t,x,u)}{\partial t} \equiv 0$ for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}$. The main idea of the controllability function method is to find a function $\Theta(x)$ ($\Theta(x) > 0$ for $x \neq 0$, $\Theta(0) = 0$) and a control $u = u(x)$ such that the following inequality holds

$$\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} \varphi_i(x, u(x)) \le -\beta \Theta^{1-\frac{1}{\alpha}}(x), \quad \beta > 0, \quad \alpha > 0. \tag{3}$$

Denote by $x(t, x_0)$ the solution of the closed-loop system $\dot{x} = \varphi(t, x, u(x))$ that satisfies the condition $x(0, x_0) = x_0$. The last inequality ensures that the trajectory of the closed-loop system steers any initial point $x_0 \in \mathbb{R}^n$ to the origin in some finite $T(x_0)$ [1] and $x(t, x_0) = 0$ for all $t \ge T(x_0)$. Moreover, the time of motion satisfies the estimate $T(x_0) \le \frac{\alpha}{\beta} \Theta^{\frac{1}{\alpha}}(x_0)$.

It is important to note that inequality (3) guaranties that the origin is stable. In this case the control $u = u(x)$ is often called a finite-time stabilizing control; and the origin is said to be a finite-time stable equilibrium [10] of system (2) with $u = u(x)$.

The paper is organized as follows. In Section 2 we consider the case $c_i = 1$, $i = 1, \dots, n-1$. Namely, we construct a class of controls $u = u(x)$ that solve the global synthesis problem for system (1). We also show that these controls satisfy the condition $|u(x)| \leq d$. In Section 3 we consider the case $\prod_{i=1}^{n-1} c_i \neq 0$. Finally, the example is given to illustrate the implementability of the approach proposed.

2. Control law construction for systems with power nonlinearity

Consider the global synthesis problem for system (1) in the case $c_i = 1$, $i = 1, \dots, n-1$. In this case system (1) takes the form

$$\begin{cases} \dot{x}_1 = u, \quad |u| \leq d, \\ \dot{x}_i = x_{i-1}, \quad i = 2, \dots, n-1, \\ \dot{x}_n = x_{n-1}^{2k+1}, \end{cases} \tag{4}$$

where $k = \frac{p}{q}$, $p > 0$ is an integer, $q > 0$ is an odd integer.

In this section we construct a controllability function and a class of bounded controls that solve the global synthesis problem for system (4).

Let us introduce the following diagonal matrices

$$D(\Theta) = \operatorname{diag}\left(\Theta^{m-1}, \Theta^{m-2}, \dots, \Theta^{m-n+1}, 1\right),$$

$$H = \operatorname{diag}\left(m-1, m-2, \dots, m-n+1, 0\right),$$

where $m = 2k(n-1) + n$.

Let $a_0 > 0$ be a fixed numbed. Suppose that F is a positive definite matrix such that the matrix $F^1 = F - FH - HF$ is positive definite. The additional conditions on a_0 and F will be obtained later.

We define the controllability function $\Theta(x)$, for $x \neq 0$, as a unique positive solution of the equation

$$2a_0\Theta^{2m} = (FD(\Theta)x, D(\Theta)x). \tag{5}$$

We remark that equation (5) has a unique positive solution, for every fixed $x \neq 0$, if the matrix F^1 is positive definite. Moreover, the function $\Theta(x)$ is continuously differentiable at every point $x \neq 0$. We complete the definition of $\Theta(x)$ by putting $\Theta(0) = 0$. Thus $\Theta(x)$ satisfies the following equality

$$2a_0\Theta^{2m}(x) = (FD(\Theta(x))x, D(\Theta(x))x). \tag{6}$$

Consider the following control law

$$u(x) = \frac{1}{\Theta^m(x)}(a, D(\Theta(x))x) + a_{n+1}\frac{x_{n-1}^{2k+1}}{\Theta^{m-1}(x)}, \tag{7}$$

where $a = (a_1, a_2, \dots, a_n)^* \in \mathbb{R}^n$. The numbers $a_i < 0$, $i = 1, \dots, n+1$ are to be chosen later.

We use the following notation

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-2} & a_{n-1} & a_n \\ 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \quad h_n = \begin{pmatrix} a_{n+1} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{8}$$

Assume that the control $u = u(x)$ of the form (7) is applied to system (4). Calculating the derivative of $\Theta(x)$ along trajectories of the closed-loop system (4), from (6) we obtain

$$\begin{aligned} \dot{\Theta}(x)\Big|_{(4)} &= \frac{\big((A^*F + FA)y(\Theta(x), x), y(\Theta(x), x)\big)}{\big((2mF - FH - HF)y(\Theta(x), x), y(\Theta(x), x)\big)} \\ &+ \frac{2\big(Fh_n, y(\Theta(x), x)\big)x_{n-1}^{2k+1}\Theta(x)}{\big((2mF - FH - HF)y(\Theta(x), x), y(\Theta(x), x)\big)}, \end{aligned} \tag{9}$$

where $y(\Theta(x), x) = D(\Theta(x))x$.

We note that since the matrix A is singular, it is impossible to choose a positive definite matrix F so that the matrix $A^*F + FA$ is negative definite. So we choose the positive definite matrix F so that the matrix $A^*F + FA$ is positive semi-definite. To this end, we consider the following Lyapunov matrix equation

$$A^*F + FA = -W, \tag{10}$$

where $W = \{w_{i,j}\}_{i,j=1}^n$ $(w_{ij} = w_{ji},\ i \neq j)$ is some positive semi-definite matrix, F is an unknown matrix.

Let us introduce the following real symmetric matrix

$$W_{n-1} = \begin{pmatrix} w_{11} & \cdots & w_{1n-1} \\ \cdots & \cdots & \cdots \\ w_{1n-1} & \cdots & w_{n-1n-1} \end{pmatrix}. \tag{11}$$

Consider the case of the positive definite matrix W_{n-1}. In [4, theorem 1] it was proved that the matrix equation (10) is solvable in the class of all positive definite matrices F if and only if the matrix W has the form

$$W = \begin{pmatrix} w_{11} & \cdots & w_{1n-1} & w_{1n-1}\frac{a_n}{a_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ w_{1n-1} & \cdots & w_{n-1n-1} & w_{n-1n-1}\frac{a_n}{a_{n-1}} \\ w_{1n-1}\frac{a_n}{a_{n-1}} & \cdots & w_{n-1n-1}\frac{a_n}{a_{n-1}} & w_{n-1n-1}\frac{a_n^2}{a_{n-1}^2} \end{pmatrix}. \tag{12}$$

Further we need the following lemma, which was proved in [4, p. 77].

Lemma 1. *The matrix W given by (12) is positive semi-definite if and only if the matrix W_{n-1} given by (11) is positive semi-definite.*

The following theorem describes the class of positive definite solutions of matrix equation (10).

Theorem 1. *Suppose that the matrices A and W are defined by (8) and (12) respectively. Furthermore, suppose that the matrix W_{n-1} defined by (11) is positive definite, and eigenvalues of the matrix*

$$A_{n-1} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{13}$$

have negative real parts. Then matrix equation (10) is solvable and its positive definite solutions have the form

$$F = \begin{pmatrix} f_{11} & \cdots & f_{1n-1} & \frac{a_n}{a_{n-1}} f_{1n-1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{1n-1} & \cdots & f_{n-1n-1} & \frac{a_n}{a_{n-1}} f_{n-1n-1} \\ \frac{a_n}{a_{n-1}} f_{1n-1} & \cdots & \frac{a_n}{a_{n-1}} f_{n-1n-1} & f_{nn} \end{pmatrix}, \tag{14}$$

where elements of the matrix $F_{n-1} = \{f_{ij}\}_{i,j=1}^{n-1}$ are defined by the matrix equation

$$A_{n-1}^* F_{n-1} + F_{n-1} A_{n-1} = -W_{n-1}$$

and $f_{nn} > 0$ is an arbitrary real number such that

$$f_{nn} > \frac{a_n^2}{a_{n-1}^2} f_{n-1n-1}. \tag{15}$$

Proof. This theorem is a simple consequence of theorem 1 and theorem 2 from [4].

Now we define the matrix F and numbers a_i, $i = 0, \ldots, n+1$ so that there exists $\beta > 0$ such that $\dot{\Theta}(x)\Big|_{(4)} \leq -\beta$. This means that inequality (3) holds for $\alpha = 1$.

Suppose that the matrix W_{n-1} is a given positive definite matrix of the form (11). Then, by Lemma 1, the matrix W of the form (12) is positive semi-definite. Suppose that the numbers $a_i < 0$, $i = 1, \ldots, n-1$ are such that the matrix A_{n-1} of the form (13) is stable, i.e. eigenvalues of the matrix A_{n-1} have negative real parts. We define the matrix F as a positive definite solution of matrix equation (10). Then, according to Theorem 1, F has the form (14).

Thus, using (9), the derivative of the controllability function takes the form

$$\dot{\Theta}(x)\Big|_{(4)} = \frac{-(Wy(\Theta(x),x),y(\Theta(x),x)) + 2(Fh_n, y(\Theta(x),x))x_{n-1}^{2k+1}\Theta(x)}{(F^1 y(\Theta(x),x), y(\Theta(x),x))}, \quad (16)$$

where $F^1 = 2mF - FH - HF$.

We introduce the following notation $I_{n,2} = \operatorname{diag}(1,\dots,1,0,0)$ is a matrix of dimension $(n\times n)$, $I_{n-1,1}$=diag $(1,\dots,1,0)$ is a matrix of dimension $(n-1)\times(n-1)$, I_{n-1} is the identity $(n-1)\times(n-1)$ matrix, $\widehat{x} = (x_1,\dots,x_{n-1})$.

Since the matrix W_{n-1} is positive definite, we have the following estimate

$$(W_{n-1}\widehat{x},\widehat{x}) \geq \lambda_{min}(\widehat{x},\widehat{x}) \quad \text{for all} \quad \widehat{x}\in\mathbb{R}^{n-1},$$

where $\lambda_{min} > 0$ is the smallest eigenvalue of the matrix W_{n-1}. Therefore,

$$-((W_{n-1} - \lambda_{min}I_{n-1})\widehat{x},\widehat{x}) - \lambda_{min}x_{n-1}^2 \leq 0 \quad \text{for all} \quad \widehat{x}\in\mathbb{R}^{n-1},$$

i.e. the matrix $W_{n-1} - \lambda_{min}I_{n-1,1}$ is positive semi-definite. Then, by Lemma 1, we have

$$-((W - \lambda_{min}I_{n,2})x,x) \leq 0 \quad \text{for all} \quad x\in\mathbb{R}^n. \quad (17)$$

Introducing the notation $b = -Fh_n$, we get

$$\begin{aligned} b_i &= -(f_{1i}a_{n+1} + \frac{a_n}{a_{n-1}}f_{in-1}), \quad i=1,\dots,n-1, \\ b_n &= a_{n+1}\frac{a_n}{a_{n-1}}f_{1n-1} + f_{nn}. \end{aligned}$$

We choose a_{n+1} so that $b_n = 0$. Thus we put

$$a_{n+1} = -\frac{f_{nn}}{f_{1n-1}}\cdot\frac{a_{n-1}}{a_n}. \quad (18)$$

Finally, we obtain

$$b_i = \left(f_{1i}\frac{f_{nn}}{f_{1n-1}} - f_{in-1}\frac{a_n^2}{a_{n-1}^2}\right)\frac{a_{n-1}}{a_n}, \quad i=1,\dots,n-1. \quad (19)$$

Combining (15) and (19), we deduce

$$b_{n-1} = \left(f_{nn} - f_{n-1n-1}\frac{a_n^2}{a_{n-1}^2}\right)\frac{a_{n-1}}{a_n} > 0.$$

Consider the following $(n-1)\times(n-1)$ matrix

$$W_{\lambda_{min}}(\Theta,x_{n-1}) = \begin{pmatrix} \lambda_{min} & 0 & \cdots & 0 & b_1\frac{x_{n-1}^k}{\Theta^{k(n-1)}} \\ 0 & \lambda_{min} & \cdots & 0 & \vdots \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_{min} & b_{n-2}\frac{x_{n-1}^k}{\Theta^{k(n-1)}} \\ b_1\frac{x_{n-1}^k}{\Theta^{k(n-1)}} & \cdots & \cdots & b_{n-2}\frac{x_{n-1}^k}{\Theta^{k(n-1)}} & 2b_{n-1} \end{pmatrix}.$$

For definiteness we assume that

$$W_{\lambda_{min}}(\Theta, x_1) = 2b_1, \quad W_{\lambda_{min}}(\Theta, x_2) = \begin{pmatrix} \lambda_{min} & b_1\dfrac{x_2^k}{\Theta^{2k}} \\ b_1\dfrac{x_2^k}{\Theta^{2k}} & 2b_2 \end{pmatrix}.$$

By direct calculation it can be shown that

$$\begin{aligned} &\lambda_{min}\big(I_{n,2}y(\Theta,x), y(\Theta,x)\big) + 2\big(b, y(\Theta,x)\big)x_{n-1}^{2k+1}\Theta = \\ &\qquad\qquad \big(W_{\lambda_{min}}(\Theta,x)\widehat{y}(\Theta,x), \widehat{y}(\Theta,x)\big), \end{aligned} \tag{20}$$

where $\widehat{y}(\Theta,x) = (x_1\Theta^{m-1}, \ldots, x_{n-2}\Theta^{m-n+2}, x_{n-1}^{k+1}\Theta^{\frac{m-n+2}{2}})$.

For $n = 2$ equality (20) reads as

$$\lambda_{min}\big(I_{2,2}y(\Theta,x), y(\Theta,x)\big) + 2\big(b, y(\Theta,x)\big)x_1^{2k+1}\Theta = 2b_1x_1^{2k+2}\Theta^m.$$

Using equality (20), we rewrite $\dot{\Theta}(x)\Big|_{(4)}$ in the form

$$\begin{aligned} \dot{\Theta}(x)\Big|_{(4)} = &-\frac{\big((W - \lambda_{min}I_{n,2})y(\Theta(x),x), y(\Theta(x),x)\big)}{(F^1y(\Theta(x),x), y(\Theta(x),x))} \\ &-\frac{\big(W_{\lambda_{min}}(\Theta(x),x_{n-1})\widehat{y}(\Theta(x),x), \widehat{y}(\Theta(x),x)\big)}{(F^1y(\Theta(x),x), y(\Theta(x),x))}, \end{aligned} \tag{21}$$

where $F^1 = 2mF - FH - HF$.

Lemma 2. *Let $\widehat{\lambda}_{min}(\Theta, x_{n-1})$ be the smallest eigenvalue of the matrix $W_{\lambda_{min}}(\Theta, x_{n-1})$. Then*

$$\widehat{\lambda}_{min}(\Theta, x_{n-1}) = \frac{1}{2}\left(\lambda_{min} + 2b_{n-1} - \sqrt{(\lambda_{min} - 2b_{n-1})^2 + 4\frac{x_{n-1}^{2k}}{\Theta^{2k(n-1)}}\sum_{i=1}^{n-2} b_i^2}\right)$$

for $n \geq 3$.

Proof. Denote by $\chi_A(\lambda)$ the characteristic polynomial of the matrix $W_{\lambda_{min}}(\Theta, x_{n-1})$. It is not difficult to establish by induction that

$$\chi_A(\lambda) = (\lambda_{min} - \lambda)^{n-3}\left(\lambda^2 - (2b_{n-1} + \lambda_{min})\lambda - \frac{x_{n-1}^{2k}}{\Theta^{2k(n-1)}}\sum_{i=1}^{n-2} b_i^2 + 2b_{n-1}\lambda_{min}\right).$$

By direct calculation, it is easy to verify that the smallest root of this equation is $\widehat{\lambda}_{min}(\Theta, x_{n-1})$. Thus the lemma is proved.

Lemma 3. *Suppose that a_0 satisfies the inequality*

$$0 < a_0 < \frac{1}{2}\lambda_{min}(F)\left(\frac{2b_{n-1}\lambda_{min}}{b_1^2 + b_2^2 + \cdots + b_{n-2}^2}\right)^{\frac{1}{k}}. \tag{22}$$

Then the matrix $W_{\lambda_{min}}(\Theta(x), x_{n-1})$ *is positive definite for every fixed* $x \neq 0$.

Proof. The matrix F is positive definite. Then, from (6), we obtain

$$2a_0\Theta^{2m}(x) \geq \lambda_{min}(F)\|y(\Theta(x), x)\|^2, \tag{23}$$

where $\lambda_{min}(F) > 0$ is the smallest eigenvalue of the matrix F. Since

$$\|y(\Theta, x)\|^2 \geq x_i^2\Theta^{2(m-i)}, \quad i = 1, \dots, n-1 \quad \text{and} \quad \|y(\Theta, x)\|^2 \geq x_n^2,$$

it follows from (22) that

$$\frac{x_i^2}{\Theta^{2i}(x)} \leq \frac{2a_0}{\lambda_{min}(F)}, \quad i = 1, \dots, n-1, \quad \frac{x_n^2}{\Theta^{2m}(x)} \leq \frac{2a_0}{\lambda_{min}(F)}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. In particular

$$\frac{x_{n-1}^2}{\Theta^{2(n-1)}(x)} \leq \frac{2a_0}{\lambda_{min}(F)}. \tag{24}$$

Combining (22) and (24), we obtain

$$\frac{x_{n-1}^{2k}}{\Theta^{2k(n-1)}(x)} < \frac{2b_{n-1}\lambda_{min}}{b_1^2 + b_2^2 + \dots + b_{n-2}^2}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. This inequality implies that

$$\widehat{\lambda}_{min}(\Theta(x), x_{n-1}) > \tfrac{1}{2}\left(\lambda_{min} + 2b_{n-1} - \sqrt{(\lambda_{min} - 2b_{n-1})^2 + 8b_{n-1}\lambda_{min}}\right)$$
$$= \tfrac{1}{2}\left(\lambda_{min} + 2b_{n-1} - \sqrt{(\lambda_{min} + 2b_{n-1})^2}\right) = 0.$$

Therefore the matrix $W_{\lambda_{min}}(\Theta(x), x_{n-1})$ is positive definite for every fixed $x \neq 0$. This concludes the proof.

First we prove that $\dot{\Theta}(x) < 0$ for any a_0 that satisfies condition (22). So suppose a_0 satisfies condition (22). Let us introduce the following notation

$$\widehat{\lambda} = \frac{1}{2}\left(\lambda_{min} + 2b_{n-1} - \sqrt{(\lambda_{min} - 2b_{n-1})^2 + 4L^k\sum_{i=1}^{n-2} b_i^2}\right),$$

where $L = \dfrac{2a_0}{\lambda_{min}(F)}$. Then, by inequality (24), we obtain that the smallest eigenvalue of the matrix $W_{\lambda_{min}}(\Theta(x), x)$ satisfies the following inequality

$$\widehat{\lambda}_{min}(\Theta(x), x_{n-1}) \geq \widehat{\lambda} > 0. \tag{25}$$

The last inequality implies that

$$\big(W_{\lambda_{min}}(\Theta(x), x_{n-1})\widehat{y}(\Theta(x), x), \widehat{y}(\Theta(x), x)\big) \geq \widehat{\lambda}\|\widehat{y}(\Theta(x), x)\|^2. \tag{26}$$

Due to positive definiteness of the matrix F^1 we have the following estimate

$$(F^1 y(\Theta(x),x), y(\Theta(x),x)) \le \lambda_{max}(F^1)\|y(\Theta(x),x)\|^2, \tag{27}$$

where $\lambda_{max}(F^1) > 0$ is the largest eigenvalue of the matrix F^1.

From (21), using (26) and (27), we obtain

$$\dot{\Theta}(x)\Big|_{(4)} \le -\frac{\big((W-\lambda_{min}I_{n,2})y(\Theta(x),x), y(\Theta(x),x)\big) + \widehat{\lambda}\cdot\|\widehat{y}(\Theta(x),x)\|^2}{\lambda_{max}(F^1)\|y(\Theta(x),x)\|^2}, \tag{28}$$

where $\widehat{y}(\Theta(x),x) = (x_1\Theta^{m-1}(x), \ldots, x_{n-2}\Theta^{m-n+2}(x), x_{n-1}^{k+1}\Theta^{\frac{m-n+2}{2}}(x))$.

Inequality (28) implies that

$$\dot{\Theta}(x)\Big|_{(4)} < 0 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Indeed, for $\|\widehat{y}(\Theta(x),x)\| \neq 0$ the last inequality is true since inequalities (17) and (25) hold. For $\|\widehat{y}(\Theta(x),x)\| = 0$, from (28), we have

$$\dot{\Theta}(x)\Big|_{(4)} \le -\frac{w_{n-1n-1}a_n^2}{\lambda_{max}(F^1)a_{n-1}^2} < 0,$$

where $\lambda_{max}(F^1) > 0$, $w_{n-1n-1} > 0$.

Thus the origin $x = 0$ is a globally asymptotically stable equilibrium of the closed-loop system (4). Now we prove that there exists $\beta > 0$ such that

$$\dot{\Theta}(x)\Big|_{(4)} \le -\beta.$$

Suppose that x_i^0, $i = 1, \ldots, n$ are real numbers such that $\sum\limits_{i=1}^{n} |x_i^0| \neq 0$. Consider a family of curves defined by

$$\begin{cases} x_1 = x_1^0|x_n^0|^{-\frac{1}{m}} sign(x_n^0)|x_n|^{\frac{1}{m}} sign(x_n), \\ x_2 = x_2^0|x_n^0|^{-\frac{2}{m}} sign(x_n^0)|x_n|^{\frac{2}{m}} sign(x_n), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{n-1} = x_{n-1}^0|x_n^0|^{-\frac{n-1}{m}} sign(x_n^0)|x_n|^{\frac{n-1}{m}} sign(x_n) \\ x_n = x_n. \end{cases} \tag{29}$$

We note that for every fixed point $x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n \setminus \{0\}$ such that $x_n^0 \neq 0$ there is exactly one curve from the family passing through x^0.

Suppose that the point $x \in \mathbb{R}^n$ lies on the curve (29) for some fixed $x^0 \neq 0$. By direct calculation, it is easy to verify that

$$\Theta(x) = \Theta(x_0)|x_n^0|^{-\frac{1}{m}}|x_n|^{\frac{1}{m}}. \tag{30}$$

Now we estimate $\dot{\Theta}(x)\Big|_{(4)}$ for every point $x \in \mathbb{R}^n$ that lies on the curve (29) with some fixed $x_0 \neq 0$. From (28), using (29) and (30), we obtain

$$\dot{\Theta}(x)\Big|_{(4)} \leq -\frac{\big((W-\lambda_{min}I_{n,2})z,z\big)+\widehat{\lambda}\cdot\left(\sum\limits_{i=1}^{n-2} z_i^2 + z_{n-1}^{2k+2}\Big(\dfrac{2a_0}{(Fz,z)}\Big)^k\right)}{\lambda_{max}(F^1)\|z\|^2}, \qquad (31)$$

where

$$z=(z_1,\ldots,z_n)=\left(\frac{x_1^0}{x_n^0}\Theta^{m-1}(x_0),\ldots,\frac{x_{n-1}^0}{x_n^0}\Theta^{m-n+1}(x_0),1\right).$$

We will show that the right-hand side of (31) is bounded from zero. Consider the function $G(\widehat{z})$ defined by

$$G(\widehat{z}) = -\frac{\big((W-\lambda_{min}I_{n,2})z,z\big)+\widehat{\lambda}\cdot\left(\sum\limits_{i=1}^{n-2} z_i^2 + z_{n-1}^{2k+2}\Big(\dfrac{2a_0}{(Fz,z)}\Big)^k\right)}{\lambda_{max}(F^1)\|z\|^2}, \qquad (32)$$

where $\widehat{z}=(z_1,\ldots,z_{n-1})$. Let R be an arbitrary number such that

$$0 < R < \frac{1}{2}\cdot\frac{a_n}{a_{n-1}}\cdot\frac{w_{n-1n-1}}{\sqrt{\sum\limits_{i=1}^{n-1} w_{in-1}^2}}. \qquad (33)$$

First we estimate the function $G(\widehat{p})$ for every point $\widehat{z}=(z_1,\ldots,z_{n-1})$ such that $z_1^2+\cdots+z_{n-1}^2 \leq R^2$. From (32) and (33) we deduce that

$$\begin{aligned} G(\widehat{z}) &= -\frac{\big((W_{n-1}-I_{n-1}\lambda_{min})\widehat{z},\widehat{z}\,\big)+\frac{a_n^2}{a_{n-1}^2}\,w_{n-1n-1}+2\frac{a_n}{a_{n-1}}\sum\limits_{i=1}^{n-1} w_{in-1}z_i}{\lambda_{max}(F^1)\|z\|^2} \\ &\leq -\frac{\frac{a_n^2}{a_{n-1}^2}\,w_{n-1n-1}-2\frac{a_n}{a_{n-1}}\sqrt{\sum\limits_{i=1}^{n-1} w_{in-1}^2}\sqrt{\sum\limits_{i=1}^{n-1} z_i^2}}{\lambda_{max}(F^1)\|p\|^2} \\ &\leq -\frac{\frac{a_n^2}{a_{n-1}^2}\,w_{n-1n-1}-2\frac{a_n}{a_{n-1}}\sqrt{\sum\limits_{i=1}^{n-1} w_{in-1}^2}\cdot R}{\lambda_{max}(F^1)(R^2+1)} \equiv -M_1(R) < 0. \end{aligned} \qquad (34)$$

Second we estimate the function $G(\widehat{z})$ for every point $\widehat{z}=(z_1,\ldots,z_{n-1})$ such that $z_1^2+\cdots+z_{n-1}^2 \geq R^2$. From (32) and (33) we deduce that

$$\begin{aligned}
G(\widehat{z}) &\leq -\frac{\widehat{\lambda}\left(z_1^2+\cdots+z_{n-2}^2+\left(\frac{2a_0}{(Fz,z)}\right)^k z_{n-1}^{2k+2}\right)}{\lambda_{max}(F^1)\|z\|^2} \\
&\leq -\frac{\widehat{\lambda}\,\min\left\{1,\left(\frac{2a_0}{\lambda_{max}(F)}\right)^k\right\}}{\lambda_{max}(F^1)}\cdot\frac{\left(\|z\|^{2k}(z_1^2+\cdots+z_{n-2}^2)+z_{n-1}^{2k+2}\right)}{\|z\|^{2k+2}} \\
&\leq -\frac{\widehat{\lambda}\,\min\left\{1,\left(\frac{2a_0}{\lambda_{max}(F)}\right)^k\right\}}{\lambda_{max}(F^1)}\cdot\frac{\left(z_1^{2k+2}+\cdots+z_{n-2}^{2k+2}+z_{n-1}^{2k+2}\right)}{\|z\|^{2k+2}} \\
&\leq -\frac{\widehat{\lambda}\,\min\left\{1,\left(\frac{2a_0}{\lambda_{max}(F)}\right)^k\right\}}{\lambda_{max}(F^1)}\cdot\frac{2^{(2-n)k}\left(z_1^2+\cdots+z_{n-2}^2+z_{n-1}^2\right)^{k+1}}{\|z\|^{2k+2}} \\
&\leq -\frac{\widehat{\lambda}\,\min\left\{1,\left(\frac{2a_0}{\lambda_{max}(F)}\right)^k\right\}}{\lambda_{max}(F^1)2^{(n-2)k}}\cdot\frac{R^{2k+2}}{(R^2+1)^{k+1}} \equiv -M_2(R) < 0. \qquad (35)
\end{aligned}$$

Thus, from (34) and (35), we obtain

$$G(\widehat{z}) \leq -\min\{M_1(R), M_2(R)\} < 0 \quad \text{for all} \quad \widehat{z}\in\mathbb{R}^{n-1}.$$

The last inequality implies that $\dot{\Theta}(x)\Big|_{(4)}$ is bounded from zero for every point $x\in\mathbb{R}^n$ such that $x_n \neq 0$. Since $\dot{\Theta}(x)\Big|_{(4)}$ is continuous at every point $x\in\mathbb{R}^n\setminus\{0\}$, we have the following estimate

$$\dot{\Theta}(x)\Big|_{(4)} \leq -\min\{M_1(R), M_2(R)\} \quad \text{for all} \quad x\in\mathbb{R}^n\setminus\{0\}. \qquad (36)$$

Thus inequality (3) is satisfied for $\alpha = 1$ and $\beta = \min\{M_1(R), M_2(R)\} > 0$. Therefore the equilibrium point $x = 0$ of the closed-loop system (4) is finite-time stable.

We proceed now to establish conditions under which the control $u = u(x)$ defined by (7) satisfies the estimate $|u(x)| \leq d$.

Lemma 4. *Suppose a_0^* is a unique positive root of the equation*

$$\sqrt{\frac{2a_0^*}{\lambda_{min}(F)}}\left(\|a\| - a_{n+1}\Big(\frac{2a_0^*}{\lambda_{min}(F)}\Big)^k\right) = d, \qquad (37)$$

where $a = (a_1,\ldots,a_n)$, $a_{n+1} < 0$, $\lambda_{min}(F) > 0$ is the smallest eigenvalue of the matrix F. If a_0 satisfies the inequality

$$0 < a_0 \leq a_0^*,$$

then the control $u = u(x)$ defined by (7) satisfies the restriction $|u(x)| \le d$ for all $x \in \mathbb{R}^n$.

Proof. Consider the function

$$\Phi(a_0) = \sqrt{\frac{2a_0}{\lambda_{min}(F)}} \left(\|a\| - a_{n+1} \Big(\frac{2a_0}{\lambda_{min}(F)} \Big)^k \right).$$

The function $\Phi(a_0)$ is continuous and strictly increasing. Moreover, $\Phi(a_0) > 0$ for all $a_0 > 0$. It is clear that

$$\Phi(0) = 0, \quad \text{and} \quad \Phi(a_0) \longrightarrow +\infty \quad \text{as} \quad a_0 \longrightarrow +\infty.$$

Then there exists a unique number $a_0^* > 0$ such that $\Phi(a_0^*) = d$.

Now we estimate the control $u = u(x)$ defined by (7). Since $0 < a_0 \le a_0^*$, using (23) and (24), we have

$$|u(x)| = \frac{\|a\| \cdot \|D(\Theta(x))\|}{\Theta(x)^m} - a_{n+1} \frac{x_{n-1}^{2k}}{\Theta^{m-n}(x)} \cdot \frac{|x_{n-1}|}{\Theta^{n-1}(x)}$$
$$\le \sqrt{\frac{2a_0}{\lambda_{min}(F)}} \left(\|a\| - a_{n+1} \Big(\frac{2a_0}{\lambda_{min}(F)} \Big)^k \right) \le \Phi(a_0^*) = d.$$

This completes the proof.

Finally, we summarize our discussion, and formulate the main result of this section. The next theorem provides a solution of the global synthesis problem for nonlinear system (4).

Theorem 2. *Suppose that the numbers $a_i < 0$, $i = 1, \dots, n-1$ are such that the matrix A_{n-1} defined by (13) is stable, a_n is an arbitrary negative number, the matrix W_{n-1} defined by (11) is an arbitrary positive definite matrix. Let the matrix F of the form (14) be a positive definite solution of equation (10) with right-hand side (12). Choose f_{nn} by (15), and a_{n+1} by (18). Furthermore, suppose that the matrix $F^1 = 2mF - FH - HF$ is positive definite. Choose a_0 such that*

$$0 < a_0 < \min \left\{ \frac{1}{2} \lambda_{min}(F) \left(\frac{2b_{n-1}\lambda_{min}}{b_1^2 + b_2^2 + \dots + b_{n-2}^2} \right)^{\frac{1}{k}}, a_0^* \right\},$$

where $\lambda_{min}(F)$ is the smallest eigenvalue of the matrix F, λ_{min} is the smallest eigenvalue of the matrix W_{n-1}, b_i is defined by (19), and a_0^* is a unique positive root of equation (37). Let the controllability function $\Theta(x)$, for every $x \in \mathbb{R}^n$, be the positive solution of equation (5). Then the control $u = u(x)$ defined by (7) solves the global synthesis problem for system (4). Moreover, the time of motion $T(x_0)$ from an arbitrary point $x_0 \in \mathbb{R}^n$ to the origin satisfies the estimate

$$T(x_0) \le \frac{1}{\min\{M_1(R), M_2(R)\}} \Theta(x_0),$$

where $M_1(R)$ and $M_2(R)$ are defined by (34) and (35) respectively.

Proof. According to (36) the inequality (3) is satisfied for $\alpha = 1$ and $\beta = \min\{M_1(R), M_2(R)\}$. Then, by theorem 1 from [2], the control $u = u(x)$ of the form (7) solves the global synthesis problem for system (4), and $T(x_0)$ satisfies the estimate

$$T(x_0) \le \frac{\alpha}{\beta}\Theta(x_0)^{\frac{1}{\alpha}} = \frac{1}{\min\{M_1(R), M_2(R)\}}\Theta(x_0).$$

Moreover, by Lemma 4, the control $u = u(x)$ satisfies the restriction $|u(x)| \le d$. This concludes the proof.

3. Global synthesis of bounded controls for systems with power nonlinearity in the case $\prod_{i=1}^{n-1} c_i \ne 0$

Now we solve the global synthesis problem for system (1) in the case c_i, $i = 1, \dots, n-1$ are some known numbers such that $\prod_{i=1}^{n-1} c_i \ne 0$. So consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_i = c_{i-1}x_{i-1}, \quad i = 2, \dots, n-1, \\ \dot{x}_n = c_{n-1}x_{n-1}^{2k+1}, \end{cases} \tag{38}$$

where $k = \frac{p}{q}$, $p > 0$ is an integer, $q > 0$ is an odd integer.

Using the results obtained in the previous section, we formulate the following theorem, which provides the solution of the global synthesis problem for nonlinear system (38).

Theorem 3. *Suppose that the conditions of Theorem 2 hold. Let the numbers $\widehat{c}_i$, $i = 1, \dots, n$ be defined by*

$$\widehat{c}_1 = 1, \quad \widehat{c}_i = c_{i-1}\widehat{c}_{i-1}, \quad i = 2, \dots, n-1, \quad \widehat{c}_n = c_{n-1}\widehat{c}_{n-1}^{\,2k+1}.$$

Let the controllability function $\Theta(x)$, for every $x \in \mathbb{R}^n$, be the positive solution of the equation

$$2a_0\Theta^{2m} = (\widehat{C}^{-1}F\widehat{C}^{-1}D(\Theta)x, D(\Theta)x), \tag{39}$$

where $\widehat{C} = \operatorname{diag}(\widehat{c}_1, \dots, \widehat{c}_n)$ is an $n \times n$ diagonal matrix. Then the control

$$u(x) = \frac{1}{\Theta^m(x)}\left(a, D(\Theta(x))\widehat{C}^{\,-1}x\right) + \frac{a_{n+1}}{\widehat{c}_{n-1}^{\,2k+1}} \cdot \frac{x_{n-1}^{2k+1}}{\Theta^{m-1}(x)} \tag{40}$$

solves the global synthesis problem for system (38). Moreover, the time of motion $T(x_0)$ from an arbitrary point $x_0 \in \mathbb{R}^n$ to the origin satisfies the estimate

$$T(x_0) \le \frac{1}{\min\{M_1(R), M_2(R)\}}\Theta(x_0), \tag{41}$$

where $M_1(R)$ and $M_2(R)$ are defined by (34) and (35) respectively.

Proof. Assume that the control $u = u(x)$ is applied to system (38). The change of variables $x_i = \widehat{c}_i z_i$, $i = 1, \ldots, n$ ($x = \widehat{C}z$, $z \in \mathbb{R}^n$) maps the closed-loop system (38) to the system

$$\begin{cases} \dot{z}_1 = v(z) \\ z_i = z_{i-1}, \quad i = 2, \ldots, n-1, \\ z_n = z_{n-1}^{2k+1}, \end{cases} \tag{42}$$

where $v(z) = u(\widehat{C}z)$. According to (39) and (40) we have

$$v(z) = u(\widehat{C}z) = \frac{1}{\widetilde{\Theta}^m(z)} \left(a, D(\widetilde{\Theta}(z))z\right) + a_{n+1} \frac{z_{n-1}^{2k+1}}{\widetilde{\Theta}^{m-1}(z)},$$

where the function $\widetilde{\Theta}(z)$, for every $z \in \mathbb{R}^n$, satisfies the equation

$$2a_0 \widetilde{\Theta}^{2m} = (FD(\widetilde{\Theta})z, D(\widetilde{\Theta})z).$$

It is clear that $\widetilde{\Theta}(z) = \Theta(\widehat{C}z)$. By Lemma 4, we deduce that the control $v(z)$ satisfies the estimate $|v(z)| \leq d$ for all $z \in \mathbb{R}^n$. This implies that the control $u(x)$ is bounded by the same constant $d > 0$ for all $x \in \mathbb{R}^n$.

Denote by $z(t, z_0)$ the solution of the closed-loop system (42) that satisfies the initial condition $z(0, z_0) = z_0$. Thus, by Theorem 2, we obtain that for every fixed $z_0 \in \mathbb{R}^n$ there exists a number $T(z_0) < +\infty$ such that $\lim\limits_{t \to T(z_0)} z(t, z_0) = 0$ and $z(t, z_0) = 0$ for all $t \geq T(z_0)$. Moreover, $T(z_0)$ satisfies the estimate

$$T(z_0) \leq \frac{1}{\min\{M_1(R), M_2(R)\}} \widetilde{\Theta}(z_0)$$

for every $z_0 \in \mathbb{R}^n$.

Denote by $x(t, x_0)$ the solution of the closed-loop system (38) that satisfies the condition $x(0, x_0) = x_0$. Since the matrix $\widehat{C}$ is nonsingular, we obtain

$$\lim_{t \to \widetilde{T}(x_0)} x(t, x_0) = 0 \quad \text{and} \quad x(t) = 0 \quad \text{for all} \quad t \geq \widetilde{T}(x_0),$$

where $\widetilde{T}(x_0) = T(\widehat{C}^{\,-1} x_0)$.

This means that the control $u = u(x)$ of the form (40) solves the global synthesis problem for system (38) and the time of motion $T(x_0)$ from an arbitrary point $x_0 \in \mathbb{R}^n$ to the origin satisfies the estimate (41). This concludes the proof.

Example 1. We solve the global synthesis problem for system (38) in the case $n = 4$, $d = 1$, $c_1 = -1$, $c_2 = \frac{1}{3}$, $c_3 = 2$, $k = 1$. So system (38) takes the form

$$\begin{cases} \dot{x}_1 = u, \quad |u| \leq 1, \\ \dot{x}_2 = -x_1, \\ \dot{x}_3 = \dfrac{1}{3} x_2, \\ \dot{x}_4 = 2x_3^3. \end{cases} \tag{43}$$

We choose negative real numbers a_1, a_2, a_3 so that the matrix A_3 defined by (13) is stable. For example, we put $a_1 = -3$, $a_2 = -3$, $a_3 = -1$. The matrix W_3 and the negative number $a_4 < 0$ may be chosen arbitrarily. We define W_3 by

$$W_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and put $a_4 = -1$. Then, according to Theorem 1, the positive definite solution of the matrix equation (10), for $f_{44} = 7$, is given by

$$F = \begin{pmatrix} \frac{11}{16} & \frac{25}{16} & \frac{1}{2} & \frac{1}{2} \\ \frac{25}{16} & \frac{25}{4} & \frac{35}{16} & \frac{35}{16} \\ \frac{1}{2} & \frac{35}{16} & \frac{49}{16} & \frac{49}{16} \\ \frac{1}{2} & \frac{35}{16} & \frac{49}{16} & 7 \end{pmatrix}.$$

Using (18), we have $a_5 = -14$.

According to (39) we define the controllability function $\Theta(x)$ as a unique positive definite solution of the equation

$$2a_0\Theta^{20} = (\widehat{C}^{\,-1}F\widehat{C}^{\,-1}D(\Theta)x, D(\Theta)x),$$

where

$$D(\Theta) = \begin{pmatrix} \Theta^9 & 0 & 0 & 0 \\ 0 & \Theta^8 & 0 & 0 \\ 0 & 0 & \Theta^7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widehat{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{2}{27} \end{pmatrix}.$$

Put $a_0 = 0.00178$. Then, by Theorem 3, the control

$$u(x) = -3\frac{x_1}{\Theta(x)} + 3\frac{x_2}{\Theta(x)^2} + 3\frac{x_3}{\Theta(x)^3} + \frac{27}{2}\frac{x_4}{\Theta(x)^{10}} + 378\frac{x_3^3}{\Theta(x)^9}$$

solves the global synthesis problem for system (43). Moreover, $u(x)$ satisfies the restriction $|u(x)| \le 1$ for all $x \in \mathbb{R}^n$.

Assume that the control $u = u(x)$ is applied to system (43). For instance, we take $x_0 = (-0.1, 0.1, -0.4, 0.3)$ as an initial point. By numerical simulation, for a solution $x(t)$ $(x(0) = x_0)$ of the closed-loop system (43), we have the following results: $\|x(100)\| = 0.051\ldots$, $\|x(5000)\| = 0.0079\ldots$, $\|x(11000)\| = 0.00064\ldots$, $\|x(15700)\| = 0.1142\ldots \times 10^{-21}$.

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Вісник Харківського національного університету імені В.Н. Каразіна
Серія "Математика, прикладна математика і механіка"
Том 81, 2015, с.52-59

Visnyk of V.N.Karazin Kharkiv National University
Ser. "Mathematics, Applied Mathematics and Mechanics"
Vol. 81, 2015, p. 52-59

УДК 510+050

On the history of our journal: 50th anniversary or 137th ordinary year of publication?

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The evolution of titles of mathematical journal Visnyk of Kharkiv University during period 1879-2015 is presented. This evolution reflects the changes of names of the University where the journal was published. We also discover and discuss several ways of numbering during the history and list names of members of Editorial Board during last 50 years.
Keywords: Visnyk, mathematical journal, Kharkiv University.

Резуненко О.В. **Щодо історії нашого журналу: 50-річний ювілей чи звичайний 137-ий рік видання?** Наведена еволюція назв математичного журналу Вісник харківського університету за період 1879-2015. Ця еволюція відображає зміни назв університету де видається журнал. Також ми розкриваємо та обговорюємо декілька способів нумерації впродовж всієї історії та наводимо призвища членів редакційної колегії за останні 50 років.
Ключові слова: Вісник, математичний журнал, Харківський університет.

Резуненко А.В. **Об истории нашего журнала: 50-летний юбилей или обычный 137-ой год издания?** Приводится эволюция названий математического журнала Вестник харьковского университета за период 1879-2015. Эта эволюция отображает изменения названий университета, где издается журнал. Также мы раскрываем и обсуждаем несколько способов нумерации в течении всей истории и приводим имена членов редакционной коллегии за последние 50 лет.
Ключевые слова: Вестник, математический журнал, Харьковский университет.
2010 Mathematics Subject Classification: 97A30, 01A55, 01A60, 01A61.

1. Introduction

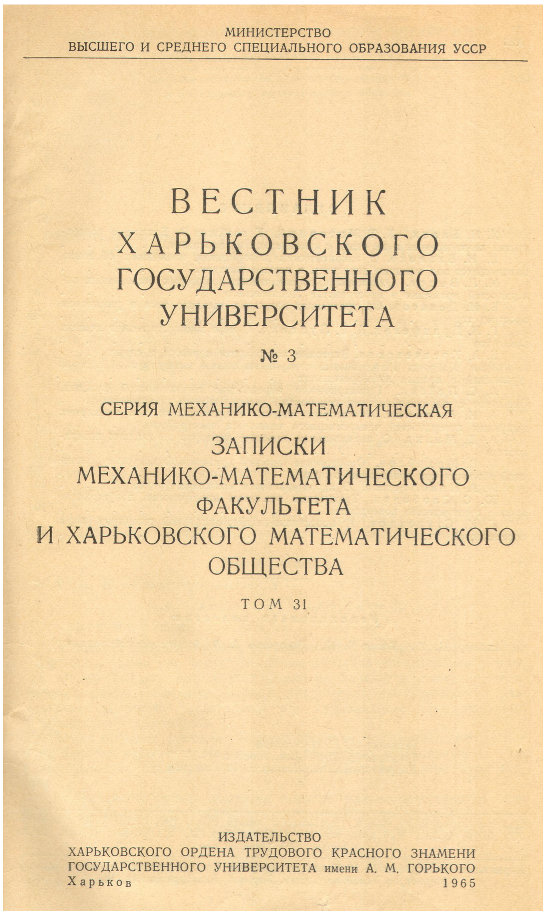
One of the goals of this note is to explain the evolution of titles of mathematical journal Visnyk of Kharkiv University. The way of numbering was also changed several times during period 1879-2015.

Although the titles were changed mainly due to evolution of names of our University, the way of numbering should be explained separately. We do it in the current issue since starting this year we will keep just the number of volumes. Another goal is to list names of mathematicians who involved in publishing process (editors).

The previous issue has the following main title: 'Visnyk of V.N.Karazin Kharkiv National University, number 1133; Ser. Mathematics, Applied Mathematics and Mechanics, (2014)' (original 'Вісник Харківського національного университету імені В.Н.Каразіна, № 1133; Серія Математика, прикладна математика і механіка, (2014)'). The number '1133' indicates the sequence number of issues of the *family* of independent journals under the family title 'Visnyk of V.N.Karazin Kharkiv National University'. The family consists of 22 series (e.g., 'Physics', 'Radiophysics and Electronics', 'Biology', etc.) where one of the series has title 'Mathematics, Applied Mathematics and Mechanics'. This way of numbering was used during 1965-2014 and most issues of our (mathematical) series had remark 'Founded in 1965' on its cover-pages. This may be considered as a proof that our journal is 50 years old. To verify this version one could find the issue dated by 1965 and check if it has number one or not. One could see 'number 3/ volume 31' on the title page.

Remark 1. *We should emphasize that words 'volume', 'number', 'issue' were mainly used completely independently. The exception is the period of 1887-1918 when volumes were composed by several issues.*

The precise title is 'Visnyk of Kharkov State University, Number 3; Ser. Mechanics and Mathematics' / 'Communications of Department of Mechanics and Mathematics and Kharkov Mathematical Society, vol. 31, (1965)' (original 'Вестник Харьковского государственного университета, № 3, Серия механико-математическая' / 'Записки механико-математического факультета и Харьковского математического общества, том 31, (1965)'). See the first scanned image below. The precise title of the second issue is 'Visnyk of Kharkov University, Number 14; Ser. Mechanics and Mathematics' / 'Communications of Department of Mechanics and Mathematics and Kharkov Mathematical Society, vol. 32, (1966)' (original 'Вестник Харьковского университета, № 14, Серия механико-математическая' / 'Записки механико-математического факультета и Харьковского математического общества, том 32, (1966)'). The third issue were numbered as 'Number 26/ volume 33, (1967)'. After two years of break, the publishing was continued with 'number/issue (year)' way of numbering and no reference to the Kharkiv Mathematical Society on the title page. It was 'number 53/issue 34 (1970)'. This way of numbering was used up to 'number 221/issue 46 (1981)'. As we can see, the number of a volume was substituted by the number of the corresponding issue. Unfortunately, starting 1982, the indication of 'issue' (as previously 'volume') was also interrupted and later continued with a gap. The general numbering from 'number 230 (1982)' to 'number 1133 (2014)' was dropped as well starting 2015. It was not the decision of our editorial board, but the one of authorities of the University. As a result, preparing the current issue we were faced to the question of how to indicate issues. In fact, all three ways 'volume/number/issue, (year)' were discarded.


МИНИСТЕРСТВО
ВЫСШЕГО И СРЕДНЕГО СПЕЦИАЛЬНОГО ОБРАЗОВАНИЯ УССР

ВЕСТНИК
ХАРЬКОВСКОГО
ГОСУДАРСТВЕННОГО
УНИВЕРСИТЕТА
№ 3

СЕРИЯ МЕХАНИКО-МАТЕМАТИЧЕСКАЯ
ЗАПИСКИ
МЕХАНИКО-МАТЕМАТИЧЕСКОГО
ФАКУЛЬТЕТА
И ХАРЬКОВСКОГО МАТЕМАТИЧЕСКОГО
ОБЩЕСТВА
ТОМ 31

ИЗДАТЕЛЬСТВО
ХАРЬКОВСКОГО ОРДЕНА ТРУДОВОГО КРАСНОГО ЗНАМЕНИ
ГОСУДАРСТВЕННОГО УНИВЕРСИТЕТА имени А. М. ГОРЬКОГО
Харьков 1965

To make decision I revised a number of issues available in our library to see the full evolution of the title/number tradition. Some selected titles and comments are presented below.

We briefly discussed above the history starting 1965. The issue dated 1964 was 'Scientific Communications. Vol. CXXXVIII / Communications of Department of Mechanics and Mathematics and Kharkov Mathematical Society. Vol. XXX, Series 4 (1964)' (original 'Ученые записки. Том CXXXVIII' / 'Записки механико-математического факультета и Харьковского математического общества. Том XXX, Серия 4 (1964)').

Remark 2. *As we described above, it was rather typical to use two subtitles for an issue. The mentioned issue published in 1964 had the first subtitle 'Scientific Communications'. The sequence of issues grouped under this subtitle was established in 1935. Initially issues of this group were the mixture of papers from different disciplines. Later the group was split into several specialized series.*

Going back in time we see that the titles were changed several times but the numbering (of the mathematical series) leads to Volume I dated by 1927. To be precise the title is (original French/Russian) 'Communications de la Société mathématique de Kharkow. Série 4. Tome I. [Сообщения Харьковского математического общества. Четвертая серия. Том I.']. See the second scanned image below. Fortunately this issue (1927) contains the explanation what does 'Série 4' mean and when the journal was established. 'The first series constitute issues 1-18. 1879-1887. The second series constitutes volumes I-XV by 6 issues each and volume XVI by issues 1-2, 1887-1918. The third series is Scientific Communications of research Chairs, mathematical department; under edition by S.M.Bernstein, I, 1924; II. 1926 and III (in press).' (original 'Первую серию составляют выпуски 1-18. 1879-1887. Вторую серию составляют том

I-XV по 6 вып. и XVI вып. 1-2, 1887-1918. Третьей серией являются Ученые записки научно-исследовательских кафедр, отдел математический; под ред. С.М.Бернштейна, I, 1924; II. 1926 и III (печатается)'). As a concluding remark we could say that we currently continue the numbering of the fourth series (started in 1927), while the very first issue appeared in 1879 and was mainly connected to the Kharkiv Mathematical Society and our University. Since 1879 the journal contains a number of mathematical papers as well as several notes on the history of Kharkiv University and Mathematical Society (see e.g., [1, 2]). It is difficult to list all famous authors who published during 137 years history of the journal. We just mention that the celebrated Stability theory has been originally published by Alexander Lyapounov in a sequence of articles in our journal (see more details e.g., [4]). It was published in 1892 as a separate edition [3] by the Kharkiv Mathematical Society (later in 1908, this edition was translated to French).

We should mention another mathematical journal published in Kharkiv 'Journal of Mathematical Physics, Analysis, Geometry' (formerly "Matematicheskaya Fizika, Analiz, Geometriya', 1994 - 2005). As our series, this journal continues the publishing tradition of the Kharkiv Mathematical Society.

Our Editorial Board traditionally sends issues to many libraries. It is interesting to note that the issue [5] in 1929 (Series 4, Vol.III) contains the list of 55 journals (Editorial Boards) which receive the printed Communications (43 journals are foreign).

More information one can find on the web-pages of our journal (currently vestnik-math.univer.kharkov.ua). All recent articles (full-text) and some scanned information on older issues are available there.

2. List of titles and editors

A. Below we list titles of *mathematical* journals/issues published by Kharkiv University (in chronological order). These titles were originally written in Russian or in French & Russian or in French & Ukrainian & Russian or in Ukrainian. Title pages of almost all issues in 1879-1940 contain titles of the journal in French. In issues where French titles are presented we do not translate them to English. We do not claim that the list below is complete since not all issues are available to check.

A1. 1879-1887. 'Communications and protocols of meetings of the Mathematical Society at Imperial Kharkov University' (original 'Сообщенiя и протоколы заседанiй Математическаго общества при Императорскомъ Харьковскомъ Университете'.

A2. 1887-1918. Original 'Communications de la Société mathématique de Kharkow. 2-e série. Tomes I-XVI. [Сообщенiя Харьковскаго математическаго общества. Вторая серiя. Тома I-XVI.]'

A3. 1927, 1928. Original 'Communications de la Société mathématique de Kharkow. Série 4. Tomes I-II. [Сообщения Харьковского математического общества. Четвертая серия. Тома I-II.]'

A4. 1929-1932. Original 'Communications de la Société mathématique de Kharkow et de l'Institute des sciences mathématiques de l'Ukraine. Série 4, t.III. [Записки Харьківського математичного товариства та Українського Інституту математичних наук. Серія 4, т. III.]'.

A5. 1933. Original 'Communications de la Société mathématique de Kharkow et de l'Institute des sciences mathématiques et mécaniques de l'Ukraine. Série 4, t.VI. [Записки Харьківського математичного товариства та Українського науково-дослідного інституту математики й механіки. Серія 4, т. VI.]'.

A6. 1934-1935. Original 'Communications de la Société mathématique de Kharkow et de l'Institute des sciences mathématiques et mécaniques de l'Université de Kharkoff. Série 4, t.VII-X. [Записки Харьківського математичного товариства та Українського науково-дослідного інституту математики й механіки при Харківському державному університеті. Серія 4, т. VII-X]'.

A7. 1936. Original 'Communications de l'Institute des sciences mathématiques et mécaniques de l'Université de Kharkoff et de la Société mathématique de Kharkoff . Série 4, t.$XIII$. [Записки науково-дослідного інституту математики й механіки при Харківському державному університеті та Харківського математичного товариства. Серія 4, т. $XIII$]'.

A8. 1938. Original 'Communications de l'Institute des sciences mathématiques et mécaniques de l'Université de Kharkoff et de la Société mathématique de Kharkoff . Série 4, t.XV_1. [Записки науково-дослідного інституту математики й механіки і Харківського математичного товариства. Серія 4, Том XV_1]'.

A9. 1938, 1940, 1948-1950. Original 'Communications de l'Institute des sciences mathématiques et mécaniques de l'Université de Kharkoff et de la Société mathématique de Kharkoff. Série 4, t.XV_2. [Записки научно-исследовательского института математики и механики и Харьковского математического общества. Серия 4, т. XV_2]'.

1940: (the same title) Series 4, Vols. XVI-XVIII. 1948: (the same) Series 4, Vol. XIX. 1949: (the same) Series 4, Vol. XXI. 1950: (the same) Series 4, Vols. XX, XXII.

A10. 1956. 'Scientific Communications. Vol. LXV / Communications of Mathematical branch of Department of Physics and Mathematics and Kharkov Mathematical Society. Vol. XXIV, Series 4 (1956)'. (Original 'Ученые записки. Том LXV // Записки математического отделения физико-математического факультета и Харьковского математического общества. Том XXIV. Серия 4').

1957: (the same) Vol. XXV. Series 4. 1960: (the same) Vol. XXVI. Series 4.

A11. 1964. 'Scientific Communications. Vol. CXXXVIII / Communications of Department of Mechanics and Mathematics and Kharkov Mathematical Society. Vol. XXX, Series 4 (1964)' (original 'Ученые записки. Том CXXXVIII' / 'Записки механико-математического факультета и Харьковского математического общества. Том XXX, Серия 4 (1964)').

A12. 1965-1967. 'Visnyk of Kharkov State University, Ser. Mechanics and Mathematics' / 'Communications of Department of Mechanics and Mathematics and Kharkov Mathematical Society' (original 'Вестник Харьковского государственного университета, Серия механико-математическая' / 'Записки механико-математического факультета и Харьковского математического общества'): Number 3 / Vol.31 (1965); Number 14 / Vol.32 (1966); Number 26 /Vol.33 (1967).

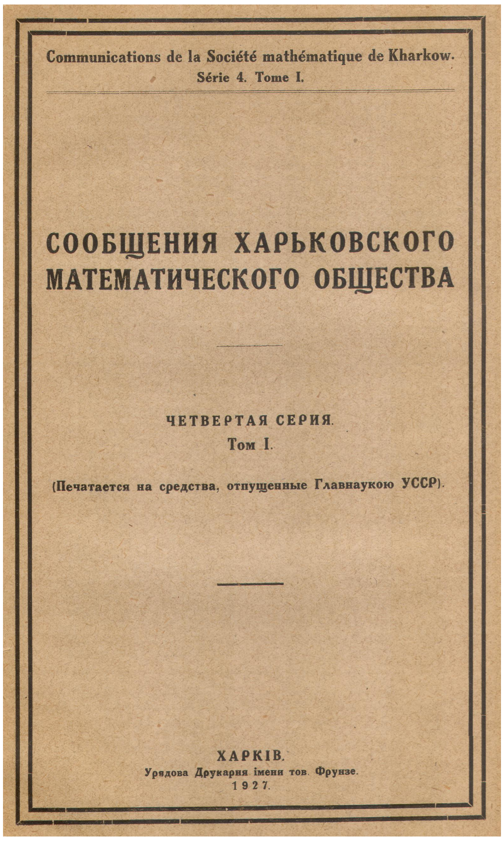
A13. 1970. 'Visnyk of Kharkov University, Number 53, Ser. Mechanics and Mathematics, issue 34' (original 'Вестник Харьковского университета, № 53, Серия механико-математическая, выпуск 34').

A14. 1973. 'Visnyk of Kharkiv University, Mathematics' (original 'Вестник Харьковского университета, Математика'). Number 93 / Issue 38 (1973).

A15. 1974-1976. 'Visnyk of Kharkiv University, Mathematics and Mechanics' (original 'Вестник Харьковского университета, Математика и механика'). Issues: Number 113 / Issue 39 (1974); Number 119 / Issue 40 (1975); Number 134 / Issue 41 (1976).

A16. 1982. 'Visnyk of Kharkiv University, Mechanics, Control theory and Mathematics physics' (original 'Вестник Харьковского университета, Механика, теория управления и математическая физика'). Number 230 (1982).

A17. 1984. 'Visnyk of Kharkiv University, Mechanics, Mathematics and Control processes' (original 'Вестник Харьковского университета, Механика, математика и процессы управления'). Number 254'84 (1984).


Communications de la Société mathématique de Kharkow.
Série 4. Tome I.

СООБЩЕНИЯ ХАРЬКОВСКОГО МАТЕМАТИЧЕСКОГО ОБЩЕСТВА

ЧЕТВЕРТАЯ СЕРИЯ.
Том I.

(Печатается на средства, отпущенные Главнаукою УССР).

ХАРКІВ.
Урядова Друкарня імени тов. Фрунзе.
1927.

A18. 1985. 'Visnyk of Kharkiv University, Control problems and Mechanics of continuous media' (original 'Вестник Харьковского университета, Проблемы управления и механики сплошных сред'). Number 277'85 (1985).

A19. 1989. 'Visnyk of Kharkiv University, Dynamical systems' (original 'Вестник Харьковского университета, Динамические системы'). Number 334'89 (1989).

A20. 1977, 1978, 1979, 1980, 1981, 1991, 1992. 'Visnyk of Kharkiv University, Applied Mathematics and Mechanics' (original 'Вестник Харьковского университета, Прикладная математика и механика').

Numbers / Issues : 148 / Issue 42 (1977); Number 174 / Issue 43 (1978);

Number 177 / Issue 44 (1979); Number 205/ Issue 45 (1980); Number 221/ Issue 46 (1981); Number 361'91 (1991); Number 361'92 (1992).

A21. 1999. 'Visnyk of Kharkiv University, Ser. Mathematics, Applied Mathematics and Mechanics' (original 'Вiсник Харкiвського унiверситету, Серiя 'Математика, прикладна математика i механiка'). *Numbers* : 444 (1999); 458 (1999).

A22. 2000-2003. 'Visnyk of Kharkiv National University, Ser. Mathematics, Applied Mathematics and Mechanics' (original 'Вiсник Харкiвського нацiонального унiверситету, Серiя 'Математика, прикладна математика i механiка'). *Numbers* : 475 (2000); 514 (2001); 542 (2002); 582 (2003).

A23. 2003-2015 (present). 'Visnyk of V.N.Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics' (original 'Вiсник Харкiвського нацiонального унiверситету iменi В. Н. Каразiна, Серiя 'Математика, прикладна математика i механiка').

Numbers : 602 (2003); 645 (2004); 711 (2005); 749 (2006); 790 (2007); 826 (2008); 850 (2009); 875 (2009); 922 (2010); 931 (2010); 967 (2011); 990 (2011); 1018 (2012); 1030 (2012); 1061 (2013); 1081 (2013); 1120 (2014); 1133 (2014).

B. Editors-in-Chief starting 1964:

N.I. Akhiezer (1964, 1966, 1967, 1970-1975), A.V. Pogorelov (1965), I.E. Tarapov (1976-1982, 1984-1989, 1991, 1992), N.A.Khizhnyak (1983), V.I. Korobov (1999-2015, present).

C. Associate Editors (members of Editorial Board) are indicated during 1964-2015:

N.I. Akhiezer (1964-1967, 1970-1975), V.V. Baranov (1976-1982, 1984, 1985), Ya.P. Blank (1965), Borisenko A.A (1999-2015), Chudinovich I.Yu (1999-2010), I.D. Chueshov (1999-2015), A. Dabrowsky (2011-2015), G.A. Dombrovsky (1992), S.Yu. Favorov (1999-2015), Yu.V. Gandel (1999-2015), A.F. Grishin (1999-2014), Yu. Karlovich (2011-2015), V.I. Korobov (1976-1982, 1984-1989, 1991, 1992, 1999-2015), Yu.I. Lubich (1976-1982, 1984, 1985, 1989, 1991), V.A. Marchenko (1964-1967, 1973-1982, 1984, 1985, 1989, 1991, 1992), N.F. Patsegon (2003-2015), A.V. Pogorelov (1964-1967, 1973-1975), A.V. Rezounenko (1999-2015), A.G. Rutkas (1992, 1999-2015), G.M. Sklyar (1999-2015), V.A. Sherbina (1999-2015), O.P. Soldatov (2011-2015), I.E. Tarapov (1976-1982, 1984-1989, 1991, 1992, 1999-2002), A.A. Yantsevich (1970-1975, 1999-2015), Zolotarev V.O. (1999-2015).

D. Responsible Editors are indicated starting 1965: A.V. Pogorelov (1965), N.I. Akhiezer (1966, 1967), A.A. Yantsevich (1970-1975), A.P. Marinich (1976-1982, 1984-1989, 1991, 1992, 1999), L.D. Stepin (1983), A.V. Rezounenko (1999-2015, present).

We should also mention the technical help in preparation of final (ready-to-print) versions of issues provided by S.V.Dmitrieva (1999-2009) and N.V.Makarova (2010-2015).

E. The total number of *mathematical articles* published during 1965-2015 is 704.

F. The presence of abstracts: 1970-2015. The presence of abstracts in *three languages* (English/Ukrainian/Russian): 2009-2015, present.

G. Full-texts PDF available online (open access): 2008-2015, present.

H. Indexed/Abstracted in Zentralblatt MATH: 1985-2015, present. Full title in Zentralblatt MATH: "Visnyk Kharkivs'kogo Universytetu. Seriya Matematyka, Prykladna Matematyka i Mekhanika". Short Title: "Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh." Documents indexed: 317 publications since 1999. Predecessor: "Vestnik Khar'kovskogo Universiteta"(documents indexed: 67 publications since 1985).

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J. Current **web-page:** http://vestnik-math.univer.kharkov.ua/ .

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