Y. Ivakhno, Big slice property in the spaces of Lipschitz functions // Visnyk of Kharkiv National University, Ser. "Mathematics, Applied Mathematics and Mechanics", N. 749 (2006), 109-118.
http://vestnik-math.univer.kharkov.ua

Вісник Харківського національного університету
Серія "Математика, прикладна математика і механіка"
УДК 517.982.22 № 749, 2006, с.109–118

Big slice property in the spaces of Lipschitz functions

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A Banach space is said to have the big slice property if every slice of its unit ball is of diameter 2. Let K be a metric space. We establish some necessary and some sufficient conditions for the big slice property of the space $\mathrm{Lip}(K)$ of all real-valued Lipschitz functions on K. In particular, $\mathrm{Lip}(K)$ has the big slice property if K is an infinite compact space. In the case of finite set K we find a characterization of extreme points of the unit ball of $\mathrm{Lip}(K)$.
2000 Mathematics Subject Classification 46B20.

1. Introduction

In this paper X stands for a Banach space, $S(X)$ and $B(X)$ are the sphere and the closed unit ball of X. Every functional $x^* \in S(X^*)$ and $\varepsilon > 0$ determine a slice of $B(X)$ by the formula

$$S(x^*, \varepsilon) = \{x \in B(X) : x^*(x) > 1 - \varepsilon\}.$$

Recall [2] that X is said to have the (diameter-) big slice property ($X \in \mathrm{BSP}$) if every slice of $B(X)$ is of diameter 2. In the other words, for every $\varepsilon > 0$ and every slice S of $B(X)$ there are x and y in S such that $\|x - y\| > 2 - \varepsilon$.

For example, $c_0 \in \mathrm{BSP}$. Every space with the Daugavet property [3] also has the BSP. If X has the Radon-Nikodym property, then it fails to have the BSP. In particular, finite dimensional spaces never enjoy the BSP.

Throughout, (K, ρ) stands for a metric space. For every pair of distinct points $t_1, t_2 \in K$ we define the slope of a function $f : K \to \mathbb{R}$ between t_1 and t_2 as

$$f(t_1; t_2) = \frac{f(t_2) - f(t_1)}{\rho(t_1, t_2)}.$$

The space of all real-valued Lipschitz functions on K will be equipped with the seminorm

$$\|f\| = \sup\{|f(t_1; t_2)| : t_1 \neq t_2 \in K\}.$$

If one quotients out the kernel of this seminorm, i.e., the constant functions, one obtains the Banach space $\mathrm{Lip}(K)$, whose norm will also be denoted by $\|\cdot\|$.

It is known [4] that in the case of a compact space K the Daugavet property for $\mathrm{Lip}(K)$ is equivalent to the locality condition on K. In this paper we investigate the big slice property for spaces of Lipschitz functions.

In the following section, "Sufficient conditions for the big slice property of $\mathrm{Lip}(K)$", it turns out that on every (infinite) compact K $\mathrm{Lip}(K)$ satisfies the big slice property. Moreover, this is also the case for every metric space which contains arbitrarily close or arbitrarily distant points. In the last section we find some necessary conditions for the big slice property of $\mathrm{Lip}(K)$.

An important tool to construct Lipschitz functions is McShane's extension theorem saying that if $M \subset K$ and $f: M \to \mathbb{R}$ is a Lipschitz function, then there is an extension of f to a Lipschitz function $F: K \to \mathbb{R}$ with the same Lipschitz constant; see [1, p.\nobreakspace {}12/13]. This will be used several times.

Remark. For a completion $\widetilde{K}$ of a metric space K $\mathrm{Lip}(K) = \mathrm{Lip}(\widetilde{K})$, consequently, we may assume without any loss of generality that K is a complete metric space.

We shall deal with the big slice property in the following equivalent form:

Lemma 1 *$X \in \mathrm{BSP}$ if and only if for every $\varepsilon > 0$ $B(X)$ is a subset of $\overline{\mathrm{co}}\{\frac{x+y}{2} : x, y \in (1+\varepsilon)\cdot B(X), \|x-y\| > 2-\varepsilon\}$.*

Proof. Necessity is easily seen by the Hahn-Banach theorem. Let us prove the sufficiency. Fix an $\varepsilon > 0$ and a slice $S(x^*, \varepsilon)$. Take a $z \in S(x^*, \delta)$ with $\delta \in (0, \varepsilon/6)$. Applying the condition, we get x_k and y_k in $(1+\delta)\cdot B(X)$ such that $\|x_k - y_k\| > 2 - \delta$ and z is approximated by a convex combination

$$\left\|z - \sum_{k=1}^{n} \lambda_k \frac{x_k + y_k}{2}\right\| < \delta.$$

Then $x^*(\sum_{k=1}^{n} \lambda_k \frac{x_k+y_k}{2}) > x^*(z) - \delta > 1 - 2\delta$ and consequently for some $j \in \{1, ..., n\}$ $x^*(\frac{x_j+y_j}{2}) > 1 - 2\delta$. Hence, $\min\{x^*(x_j), x^*(y_j)\} > 2 - 4\delta - \max\{x^*(x_j), x^*(y_j)\} \geq 2 - 4\delta - (1+\delta) = 1 - 5\delta$. Let $x = \frac{x_j}{\|x_j\|}$, $y = \frac{y_j}{\|y_j\|}$. These elements belong to $S(x^*, \varepsilon)$ since $\min\{x^*(x), x^*(y)\} > \frac{1-5\delta}{1+\delta} > \frac{6-5\varepsilon}{6+\varepsilon} > 1-\varepsilon$. On the other hand, $\|x-y\| = \|\frac{x_j}{\|x_j\|} - \frac{y_j}{\|y_j\|}\| \geq \|x_j - y_j\| - \|x_j\cdot(1-\frac{1}{\|x_j\|})\| - \|y_j\cdot(1-\frac{1}{\|y_j\|})\| > 2 - \delta - 2\cdot(1+\delta)\cdot(1-\frac{1}{1+\delta}) = 2 - 3\delta > 2 - \varepsilon$, as needed. □

Observe that this lemma is still true even if the elements x, y are considered as elements of $B(X)$. Such pairs of elements will be used in the further conclusions repeatedly. Let us fix a special notation for short:

Definition 1 *For a pair of elements $x, y \in B(X)$ the expression $\frac{x+y}{2}$ will be called an ε-arithmetic mean if $\|x-y\| > 2-\varepsilon$.*

2. Sufficient conditions for the big slice property of $\mathrm{Lip}(K)$

The following technical lemma will be used in this section several times.

Lemma 2 *Assume that for every $\varepsilon > 0$ there are two systems of points $\{t_n\}_{n=1}^{\infty}$, $\{\tau_n\}_{n=1}^{\infty} \subset K$ (write $\rho(t_n, \tau_n) = \rho_n$) and numbers $R_n > \rho_n > r_n > 0$ with*

$$0 < \frac{2\rho_n}{R_n - \rho_n} \le \varepsilon, \quad 0 < \frac{2r_n}{\rho_n - r_n} \le \varepsilon \tag{1}$$

and let the system of rings $\{B(t_n, R_n)\backslash B(t_n, r_n)\}_{n=1}^{\infty}$ be disjoint. Then $\mathrm{Lip}(K)$ *has the big slice property.*

Proof. Let us show that the condition of lemma 1 is satisfied. Take any $\varepsilon > 0$ and $f \in B(\mathrm{Lip}(K))$. For every $k \in \mathbb{N}$ define

$$x_k(t) = f(t) \text{ when } t \notin B(t_k, R_k)\backslash B(t_k, r_k).$$

Besides, take $x_k(\tau_k) = f(t_k) + \rho_k$. So, $x_k(\cdot)$ is defined on a subset of K. Let us estimate the norm $\|x_k\| = \sup |x_k(t;\tau)|$. There are three possibilities on the disposal of t, τ.

If neither t nor τ equals τ_k, then $|x_k(t;\tau)| = |f(t;\tau)| \le \|f\| \le 1$. Let one of the points, say τ, be τ_k and assume $t \notin B(t_k, R_k)$. Then

$$|x_k(t;\tau)| = \frac{|f(t_k) + \rho_k - f(t)|}{\rho(t, \tau_k)} \le \frac{\|f\| \cdot \rho(t, t_k) + \rho_k}{\rho(t, \tau_k)}$$

$$\le \frac{\rho(t, \tau_k) + 2\rho_k}{\rho(t, \tau_k)} \le 1 + \frac{2\rho_k}{R_k - \rho_k} \le 1 + \varepsilon.$$

In the last case let $\tau = \tau_k$, $t \in B(t_k, r_k)$. Then again

$$|x_k(t;\tau)| \le \frac{\|f\| \cdot \rho(t, t_k) + \rho_k}{\rho(t, \tau_k)} < \frac{r_k + \rho_k}{\rho_k - r_k} = 1 + \frac{2r_k}{\rho_k - r_k} \le 1 + \varepsilon.$$

So, $\|x_k\| \le 1 + \varepsilon$. Extend x_k to a function on K preserving the Lipschitz constant, still denoted by x_k. Build $y_k(t)$ by the same scheme with $y_k(\tau_k) = f(t_k) - \rho_k$. This results in the following inequality:

$$\|x_k - y_k\| \ge \frac{(x_k - y_k)(\tau_k) - (x_k - y_k)(t_k)}{\rho(t_k, \tau_k)} = \frac{2\rho_k}{\rho_k} = 2.$$

Now take any $n \in \mathbb{N}$ and observe that the last sum of

$$\left\| f - \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{x_k + y_k}{2} \right\| = \frac{1}{n} \cdot \left\| \sum_{k=1}^{n} \left(f - \frac{x_k + y_k}{2}\right) \right\|$$

is nonzero only in the rings $B(t_k, R_k)\backslash B(t_k, r_k)$, where it takes the values of the corresponding summands. Therefore,

$$\Big\| f - \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{x_k + y_k}{2} \Big\| \le \frac{\|f\| + 2\cdot(1+\varepsilon)}{n} \le \frac{4}{n} \longrightarrow 0$$

as $n \to \infty$. So, f is approximated by convex combinations of elements of the form $\frac{x+y}{2}$, as we needed for lemma 1. $\mathrm{Lip}(K) \in \mathrm{BSP}$. □

Now we are able to establish the sufficient conditions on K for the big slice property of $\mathrm{Lip}(K)$ which were announced above.

Theorem 1 *If* $\inf\{\rho(t,\tau) : t \neq \tau \in K\} = 0$, *then* $\mathrm{Lip}(K) \in \mathrm{BSP}$.

Proof. For every $\varepsilon > 0$ we shall construct a system of points satisfying the condition of lemma 2. Take distinct points t_n and τ_n (write $\rho(t_n, \tau_n) = \rho_n$) such that $\rho_n \to 0$. Define R_n and r_n by

$$\frac{2\rho_n}{R_n - \rho_n} = \varepsilon, \quad \frac{2r_n}{\rho_n - r_n} = \varepsilon$$

and observe that these values also tend to zero.

Now in connection with the structural properties of the system of sets $B(t_n, R_n)$ we have to split two opposite cases. Let us consider the first one: assume that for every subsequence $\{t_k\}_{k\in I} \subset \{t_n\}$ a $k_0 \in I$ can be found such that only finite set of indices $k \in I$ give

$$\rho(t_{k_0}, t_k) \le R_{k_0} + R_k.$$

Then the following inductive procedure provides us with the required subsequence. Applying the assumption, take $t_{k_1} \in \{t_n\}_{n=1}^{\infty}$ and sufficiently large $n_1 \ge k_1$ such that $\rho(t_{k_1}, t_k) > R_{k_1} + R_k$ for any $k > n_1$. Then take $t_{k_2} \in \{t_n\}_{n=n_1+1}^{\infty}$ and $n_2 \ge k_2$ such that $\rho(t_{k_2}, t_k) > R_{k_2} + R_k$ for all $k > n_2$. Continuing in this manner, we build a subsequence $\{t_{k_1}, t_{k_2}, \dots\}$ satisfying for all $i \neq j$

$$\rho(t_{k_i}, t_{k_j}) > R_{k_i} + R_{k_j}.$$

It obviously implies that the system of rings $B(t_{k_n}, R_{k_n})\backslash B(t_{k_n}, r_{k_n})$ is disjoint. The condition of lemma 2 is satisfied, hence, $\mathrm{Lip}(K)$ has the big slice property.

Consider the second case: let $N \subset \mathbb{N}$ be an infinite subsequence such that every $n \in N$ defines an infinite $N_n \subset N$ giving for all $k \in N_n$

$$\rho(t_n, t_k) \le R_n + R_k. \tag{2}$$

By the infinity of N and since $R_n \to 0$ we can find $n_1 \in N$ such that the condition (2) for all $k \in N_{n_1}$ holds simultaneously with the inequality $R_{n_1} < 1/2$. Since N_{n_1} is also infinite, it contains an n_2 also satisfying (2) for all $k \in N_{n_2}$ simultaneously with $R_{n_2} < 1/4$. Further, by the infinity of N_{n_2} it contains n_3 such that for all $k \in N_{n_3}$ (2) holds together with $R_{n_3} < 1/8$. Continuing in this manner, we find an infinite sequence t_{n_k}. This sequence is fundamental because

$$\rho(t_{n_k}, t_{n_p}) \le (R_{n_k} + R_{n_{k+1}}) + (R_{n_{k+1}} + R_{n_{k+2}}) + \cdots + (R_{n_{p-1}} + R_{n_p})$$
$$< \frac{2}{2^k} + \frac{2}{2^{k+1}} + \cdots + \frac{2}{2^{p-1}} < \frac{1}{2^{k-2}} \to 0$$

as $k \to \infty$. Since K is a complete space, t_{n_k} converges to some $t \in K$. Thereby, the simplest way to obtain a disjoint system of rings is to pass to a system having a common center in the point t. Let τ_1' be an arbitrary element t_{n_k}; write $\rho_1' = \rho(t, \tau_1')$ and take R_1' and r_1' such that the inequalities (1) for these values hold. Define a ring $B_1' = B(t, R_1') \backslash B(t, r_1')$. In order to define the next ring take any positive $R_2' \le r_1'$ and k such that $\rho(t, t_{n_k}) = \rho_2'$ satisfy (1) together with sufficiently little r_2'. Write $\tau_2' = t_{n_k}$ and $B_2' = B(t, R_2') \backslash B(t, r_2')$. Continuing this process, we obtain a sequence of disjoint rings as needed in lemma 2. Lip(K) has the big slice property. □

Corollary 1 *For a compact space K Lip(K). has the BSP if and only if K is infinite.*

Theorem 2 *If K is unbounded , i.e., $\sup\{\rho(t,\tau) : t \ne \tau \in K\} = \infty$, then* Lip($K$) $\in$ BSP.

Proof. The unboundedness condition implies that for every $t \in K$ and every $r > 0$ there is $\tau \in K$ such that $\rho(t, \tau) > r$.

Given an $\varepsilon > 0$, we build again a system of rings in order to apply lemma 2. Take any $t \in K$ and write $t_n = t$; also take any $r_1 > 0$. Applying the unboundedness of K, find $\tau_1 \in K$ (write $\rho(t_1, \tau_1) = \rho_1$) and $R_1 > 0$ such that the inequalities (1) hold for $n = 1$. Then write $r_2 = R_1$ and find $\tau_2 \in K$ ($\rho(t_2, \tau_2) = \rho_2$) and $R_2 > 0$ such that the inequalities hold for $n = 2$. Continuing similarly, we build a disjoint sequence of rings $B(t_n, R_n) \backslash B(t_n, r_n)$. Lemma 2 can be applied; Lip(K) has the big slice property. □

Consequently, the cases when Lip(K) fails the BSP are contained in the cases of spaces K such that the "limiting distances" $\sup_{t \ne \tau} \rho(t, \tau)$ and $\inf_{t \ne \tau} \rho(t, \tau)$ are bounded from both zero and infinity. But even in this case Lip(K) may have the big slice property.

Example. Let K be an arbitrary infinite set. Introduce the metric ρ on K by taking $\rho(t, \tau) = C > 0$ for all $t \ne \tau$. Then Lip(K) has the big slice property. *Proof.* $\rho(t, \tau)$ is constant for all $t \ne \tau$, so we may assume that $C = 1$. Fix an $f \in B(\text{Lip}(K))$ and write $\alpha_0 = \inf_K f(t)$. Now take any disjoint sequence $t_1, \tau_1, t_2, \tau_2, \dots \in K$ and define for all indices k

$$x_k(t_k) = \alpha_0, \quad x_k(\tau_k) = \alpha_0 + 1, \quad \text{and } x_k(t) = f(t) \text{ for } t \notin \{t_k, \tau_k\}.$$

Correspondingly,

$$y_k(t_k) = \alpha_0 + 1, \quad y_k(\tau_k) = \alpha_0, \quad \text{and } y_k(t) = f(t) \text{ for the rest of } K.$$

Obviously, x_k and y_k lie in $B(\mathrm{Lip}(K))$ and $\|x_k - y_k\| = 2$. Moreover, f is approximated by convex combinations of elements $\frac{x_k+y_k}{2}$:

$$\Big\| f - \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{x_k + y_k}{2} \Big\| = \frac{1}{n} \cdot \Big\| \sum_{k=1}^{n} (f - \frac{x_k + y_k}{2}) \Big\| \le \frac{2}{n} \longrightarrow 0,$$

as we needed to apply lemma 1. □

A less trivial example can be constructed by means of discrete subsets of l_∞. Define for a natural n

$$K = \{(x_1, x_2, \dots) \in l_\infty : \text{all } x_k \in \{1, \dots, n\}\}. \tag{3}$$

If $n = 2$, K is a discrete metric space like in the example above, so $\mathrm{Lip}(K) \in$ BSP.

If $n = 3$, the space $\mathrm{Lip}(K)$ again possesses the big slice property, but the proof becomes much more complicated.

Question. Does $\mathrm{Lip}(K)$ have the big slice property for spaces K of the form (3) with any n?

3. Necessary conditions for the big slice property of $\mathrm{Lip}(K)$

Theorem 3 *Under the following condition on K $\mathrm{Lip}(K)$ fails the BSP:*

There is an $f \in S(\mathrm{Lip}(K))$, a sequence $\{t_1, s_1, \dots, t_m, s_m\} \in K$ with $f(t_i; s_i) = 1$ for $i = 1, \dots, m$, and an $\varepsilon > 0$ such that every ε-arithmetic mean $\frac{x+y}{2}$ satisfies for some i

$$\frac{x+y}{2}(t_i; s_i) < 1 - \varepsilon.$$

Proof. Assume to the contrary that $\mathrm{Lip}(K) \in$ BSP. Then we can approximate f by a convex combination of some $\frac{\varepsilon}{m}$-arithmetic means:

$$\Big\| f - \sum_{k=1}^{n} \lambda_k \cdot \frac{x_k + y_k}{2} \Big\| < \frac{\varepsilon}{m}.$$

In particular,

$$\frac{\varepsilon}{m} > \Big(f - \sum_{k=1}^{n} \lambda_k \cdot \frac{x_k + y_k}{2} \Big)(t_i; s_i) = f(t_i; s_i) - \sum_{k=1}^{n} \lambda_k \cdot \frac{x_k + y_k}{2}\,(t_i; s_i).$$

Consequently,

$$\sum_{k=1}^{n} \lambda_k \cdot \frac{x_k + y_k}{2}\,(t_i; s_i) > 1 - \frac{\varepsilon}{m}. \tag{4}$$

On the other hand, by assumption on $\{t_i, s_i\}$, for every k there is an i such that

$$\frac{x_k + y_k}{2}(t_i; s_i) < 1 - \varepsilon. \tag{5}$$

For every i define A_i as the set of all indices k for which (5) holds; write $\alpha_i = \sum_{k\in A_i} \lambda_k$. Observe that among the α_i's there is at most one $\alpha_{i_0} \geq \frac{1}{m}$ (otherwise $1 = m\cdot\frac{1}{m} > \sum_{i=1}^m \alpha_i \geq \sum_{k=1}^n \lambda_k = 1$). Therefore,

$$\sum_{k=1}^n \lambda_k \cdot \frac{x_k + y_k}{2}(t_{i_0}; s_{i_0}) < \sum_{k\in A_{i_0}} \lambda_k \cdot (1-\varepsilon) + \sum_{k\notin A_{i_0}} \lambda_k \cdot 1$$

$$= \sum_{k=1}^n \lambda_k - \varepsilon\cdot \sum_{k\in A_{i_0}} \lambda_k \leq 1 - \frac{\varepsilon}{m},$$

which contradicts (4). □

Remark. Observe that there is no need to define the function f on the whole space K. All conditions on f, in fact, refer to the properties of $\mathrm{Lip}(\{t_1, s_1, \ldots, t_m, s_m\})$.

This idea is developed in the following theorem 4.

Theorem 4 *Under the following condition on K $\mathrm{Lip}(K)$ fails the BSP:*

There is an $\varepsilon > 0$, a finite subset $M \subset K$, and an extreme point f of $B(\mathrm{Lip}(M))$ such that every ε-arithmetic mean $\frac{x+y}{2}$ of $\mathrm{Lip}(K)$ satisfies

$$\left\| f - \frac{x+y}{2}\Big|_M \right\|_{\mathrm{Lip}(M)} > \varepsilon.$$

In order to prove this theorem we establish a characterization of extreme points of $B(\mathrm{Lip}(M))$ at first.

Lemma 3 *Let M be a finite metric space, $f \in B(\mathrm{Lip}(M))$. f is an extreme point of $B(\mathrm{Lip}(M))$ if and only if for every pair of distinct points $t, \tau \in M$ there is a sequence $s_0, \ldots, s_n \in M$ with $s_0 = t$, $s_n = \tau$ such that $|f(s_{i-1}; s_i)| = 1$ for all $i = 1, \ldots, n$.*

Proof. We prove sufficiency at first. Assume that $f = \frac{g+h}{2}$ for some $g, h \in B(\mathrm{Lip}(M))$ and let us prove that $g = f = h$. It suffices to prove that $g(t;\tau) = f(t;\tau)$ for all $t, \tau \in M$. Let $g(t;\tau) \neq f(t;\tau)$ and find the corresponding $s_0, \ldots, s_n$. Since

$$\sum_{i=1}^n [(f-g)(s_i) - (f-g)(s_{i-1})] = (f-g)(\tau) - (f-g)(t) \neq 0,$$

$g(s_{i-1}; s_i) \neq f(s_{i-1}; s_i) = \pm 1$ for some i. Denote s_{i_0-1} and s_{i_0} as s, s' in such order that $f(s; s') = +1$. Then $g(s; s') < 1$, since $\|g\| \leq 1$. Therefore, by the equality

$f = \frac{g+h}{2}$, $h(s;s') > 1$ and hence $\|h\| > 1$, which is not possible. Consequently, $g(t;\tau) = f(t;\tau)$. f is an extreme point.

In order to prove the necessity assume that for some $t \neq \tau \in M$ the condition is not satisfied. Consider the set

$$A = \{\tau\} \cup \{s \in M : \text{there are } s{=}s_0, s_1, \ldots, s_n{=}\tau \text{ with } |f(s_{i-1}; s_i)| = 1\ \forall i\}.$$

$A \neq M$, since $t \notin A$. Define

$$\alpha_1 = \sup\{\alpha \geq 0 : \|f + \chi_A \cdot \alpha\| \leq 1\},$$
$$\alpha_0 = \inf\{\alpha \leq 0 : \|f + \chi_A \cdot \alpha\| \leq 1\}.$$

Observe that $\alpha_1 \neq 0$. Indeed, otherwise for every $\alpha > 0$ $\|f + \chi_A \cdot \alpha\| > 1$, i.e., $|(f + \chi_A \cdot \alpha)(u; s)| > 1$ for some $u \notin A$, $s \in A$. Passing to the limit as $\alpha \to 0$ and applying the fact that M is finite, find $u \notin A$ and $s \in A$ such that $|f(u;s)| = 1$. But it means that $u \in A$, which is a contradiction. By the same argument $\alpha_0 \neq 0$.

Then, for $\alpha = \min\{|\alpha_0|, |\alpha_1|\} > 0$

$$\|f + \chi_A \cdot \alpha\| \leq 1 \text{ and } \|f - \chi_A \cdot \alpha\| \leq 1.$$

This implies that f is not an extreme point of $B(\mathrm{Lip}(M))$, because $f = \frac{f + \chi_A \cdot \alpha + f - \chi_A \cdot \alpha}{2}$. □

Proof of theorem 4. Denote $r = \inf_M \rho(t,\tau) > 0$, $R = \sup_M \rho(t,\tau) < \infty$. The theorem will be proved by applying theorem 3. Let $\frac{x+y}{2}$ be any $\frac{\varepsilon \cdot r}{|M| \cdot R}$-arithmetic mean. In particular, it is also an ε-arithmetic mean, so, by the condition there are $s, s' \in M$ such that

$$\Big(f - \frac{x+y}{2}\Big)(s') - \Big(f - \frac{x+y}{2}\Big)(s) > \varepsilon \cdot \rho(s,s') > \varepsilon \cdot r. \tag{6}$$

Applying lemma 3 find in M $s_0{=}s, s_1, \ldots, s_n{=}s'$ such that $|f(s_{i-1}; s_i)| = 1$ for every $i = 1, \ldots, n$. We may also assume that $s_i \neq s_j$ when $i \neq j$, so $n < |M|$.

Let us show that for some $i_0 \in \{1, \ldots, n\}$

$$\Big|\Big(f - \frac{x+y}{2}\Big)(s_{i_0}) - \Big(f - \frac{x+y}{2}\Big)(s_{i_0 - 1})\Big| > \frac{\varepsilon \cdot r}{|M|}.$$

Otherwise

$$\Big|\Big(f - \frac{x+y}{2}\Big)(s') - \Big(f - \frac{x+y}{2}\Big)(s)\Big| \leq \sum_{i=1}^{n} \Big|\Big(f - \frac{x+y}{2}\Big)(s_i) - \Big(f - \frac{x+y}{2}\Big)(s_{i-1})\Big|$$
$$\leq n \cdot \frac{\varepsilon \cdot r}{|M|} < |M| \cdot \frac{\varepsilon \cdot r}{|M|} = \varepsilon \cdot r,$$

which contradicts (6). Denote $s_{i_0 - 1}$ and s_{i_0} as t, τ in such order that $f(t;\tau) = +1$. Then

$$\frac{\varepsilon \cdot r}{|M|} < \Big|\Big(f - \frac{x+y}{2}\Big)(\tau) - \Big(f - \frac{x+y}{2}\Big)(t)\Big|$$

$$= \rho(t,\tau)\cdot\left|f(t;\tau) - \frac{x+y}{2}(t;\tau)\right| \le R\cdot\left|1 - \frac{x+y}{2}(t;\tau)\right|.$$

And hence

$$\frac{x+y}{2}(t;\tau) < 1 - \frac{\varepsilon\cdot r}{|M|\cdot R}, \tag{7}$$

as we needed to apply theorem 3. $\mathrm{Lip}(K)$ fails the big slice property. □

Let us construct an example of a space $\mathrm{Lip}(K) \notin \mathrm{BSP}$ on the base of theorem 4.

At first we have to establish some general properties of ε-arithmetic means.

Consider an ε-arithmetic mean $\frac{x+y}{2}$. Since $\|x-y\| > 2-\varepsilon$, there are $a, b \in K$ such that

$$2 - \varepsilon < (x-y)(a;b) = x(a;b) - y(a;b).$$

Therefore,

$$x(a;b) > 2 - \varepsilon + y(a;b) \ge 1 - \varepsilon; \tag{8}$$

$$y(b;a) > 2 - \varepsilon - x(a;b) \ge 1 - \varepsilon. \tag{9}$$

Finally,

$$\frac{x+y}{2}(a;b) = \frac{x(a;b)+y(a;b)}{2} < \frac{1-(1-\varepsilon)}{2} = \frac{\varepsilon}{2}. \tag{10}$$

Example. Define the sets $M = \{t,\tau\}$, $K = M \cup \{s_1, s_2, \dots\}$ and the metric ρ which takes the following values on K:

$$\rho(t,\tau) = 2, \quad \rho(s_n, s_k) = 1, \quad \rho(t, s_n) = \rho(\tau, s_n) = 1$$

for all numbers n, k. The function f will be defined on M as

$$f(t) = -1, \quad f(\tau) = 1.$$

Now, let $\frac{x+y}{2}$ be an arbitrary ε-arithmetic mean. Let us show that $\varepsilon < |(f - \frac{x+y}{2})(t;\tau)| = |1 - \frac{x+y}{2}(t;\tau)| = 1 - \frac{x+y}{2}(t;\tau)$, i.e., that

$$\frac{x+y}{2}(\tau) - \frac{x+y}{2}(t) < 2\cdot(1-\varepsilon).$$

Take any of the elements $a, b \in K$ providing $(x-y)(a;b) > 2-\varepsilon$. There are tree cases on their disposal.

In the first case assume that both a and b lie in $\{s_1, s_2, \dots\}$. Then

$$\frac{x+y}{2}(\tau) - \frac{x+y}{2}(t) = \frac{1}{2}\cdot\Big(x(\tau) - x(t) + y(\tau) - y(t)\Big)$$

$$= \frac{1}{2}\cdot\Big(x(\tau)-x(a)+x(a)-x(b)+x(b)-x(t)+y(\tau)-y(b)+y(b)-y(a)+y(a)-y(t)\Big),$$

and due to the inequalities (8) and (9)

$$\le \frac{1}{2}\cdot(1 - 1 + \varepsilon + 1 + 1 - 1 + \varepsilon + 1) = 1 + \varepsilon < 2 - 2\varepsilon$$

if $\varepsilon < 1/3$. In the second case let $a \in M$, $b \in \{s_1, s_2, \dots\}$ (or contrariwise). Then by the triangle inequality and equation (10)

$$\frac{x+y}{2}(\tau) - \frac{x+y}{2}(t) < \frac{\varepsilon}{2} \cdot 1 + \left\| \frac{x+y}{2} \right\| \cdot 1 = 1 + \frac{\varepsilon}{2} < 2 - 2\varepsilon$$

if $\varepsilon < 2/5$. In the last case $a = t$, $b = \tau$ (or contrariwise). Then by the equation (10)

$$\frac{x+y}{2}(\tau) - \frac{x+y}{2}(t) < \frac{\varepsilon}{2} < 2 - 2\varepsilon$$

if $\varepsilon < 4/5$. In any case we may apply Theorem 4 deducing that $\mathrm{Lip}(K)$ fails the big slice property.

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