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# The Justification of Numerical Solution of Boundary Integral Equations of Wave Scattering Problems on Impedance Lattice

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V. D. Dushkin. **The Justification of Numerical Solution of Boundary Integral Equations of Wave Scattering Problems on Impedance Lattices.** The justification of the method for numerical solution of the boundary integral equations of the problems of waves scattering of on the impedance lattice had been proposed. The convergence of the approximate solutions to the exact solution had been proved.
*Keywords:* the boundary integral equations, the speed of convergence, the polynomials of Chebyshev.

Душкін В. Д. **Обгрунтування чисельного розв'язку граничних інтегральних рівнянь задач розсіювання хвиль на імпедансній стрічці.** Дано обґрунтування методу чисельного розв'язку систем граничних інтегральних рівнянь задачі розсіювання хвиль на імпедансній стрічці. Доведено збіжність послідовності наближених розв'язків до точного розв'язку.
*Ключові слова:* граничні інтегральні рівняння, швидкість збіжності процесу наближень, многочлени Чебишева.

Душкин В. Д. **Обоснование численного решения граничных интегральных уравнений задач рассеивания волн на импедансной ленте.** Дано обоснование метода численного решения систем граничных интегральных уравнений задачи рассеивания волн на импедансной ленте. Доказана сходимость процесса приближений к точному решению задачи.
*Ключевые слова:* граничные интегральные уравнения, скорость сходимости, многочлены Чебышева.

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## 1. Introduction.

The process of developing of some mathematical models by using the method of integral operators' parametric representations [1, 2] leads to the consideration of equations of two different types.

Equations of the first type are singular integral equations of the first kind:

$$\int_{-1}^{1} \frac{1}{\tau-\xi}\frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}} + \frac{1}{\pi}\int_{-1}^{1} Q(\xi,\tau)\frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}} -$$

$$-\frac{c_1}{\pi}\int_{-1}^{\xi} \frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}} = f(\xi), \quad |\xi|<1; \tag{1}$$

with the additional condition

$$\int_{-1}^{1} \frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}} = 0. \tag{2}$$

The equations of the second type are Fredholm equations of the second kind:

$$\vartheta(\xi) - \frac{c_2}{\pi}\sqrt{1-\xi^2}\int_{-1}^{1} \ln|\tau-\xi|\frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}} +$$

$$+\frac{1}{\pi}\int_{-1}^{1} K(\xi,\tau)\frac{\vartheta(\tau)d\tau}{\sqrt{1-\tau^2}} = g(\xi), \quad |\xi|<1. \tag{3}$$

In the equations (1)-(3) it is assumed that

$$f \in C^{\mu,\gamma}[-1,1], \quad Q \in C^{\mu,\gamma}([-1,1]\times[-1,1]), \quad \gamma>0, \tag{4}$$

$$g \in C^{0,\psi}[-1,1], \quad K \in C^{0,\psi}([-1,1]\times[-1,1]), \quad 0<\psi\le\frac{1}{2}. \tag{5}$$

In particular, equation (1)-(3) occurred in considering of the wave diffraction problem on the system of superconducting tapes [3, 4]. The scheme of discretization of these equations had been proposed in [4]. It is based on the method of discrete singularities [5] . Using of this scheme numerical experiments had been conducted.

Unfortunately, the proof of convergence approximate solutions of the equations (1)-(3) to the exact solution of these equations had not been given.

In this article the proof of convergence of the approximation process is offered.

Also the estimation of convergence rate of approximate solutions to the exact solution of the equations had been given.

## 2. The functional spaces and integral operators.

Let us consider the Lebesgue spaces $L^2_\alpha$, $\alpha = \pm\frac{1}{2}$, of measurable functions with the inner product

$$(\nu,\nu)_\alpha = \int\limits_{-1}^{1} u(\tau)\cdot\overline{\nu}(\tau)(1-\tau^2)^\alpha d\tau \tag{6}$$

and norm $\|\nu\|_\alpha = \sqrt{(\nu,\nu)_\alpha}$.

We introduce the subspaces $\Pi_{n,\alpha}$ of spaces $L^2_\alpha$ . Elements of these subspaces are polynomials of degree $n$. Also we define spaces $L^2_{0,\alpha} = \left\{u \in L^2_\alpha | (u,1)_\alpha = 0\right\}$ and $\Pi^0_{n,\alpha} = \{u \in \Pi_{n,\alpha} | (u,1)_\alpha = 0\}$ .

We introduce the operators (see [6, 7]):

$$\Lambda : L^2_{-\frac{1}{2}} \to L^2_{-\frac{1}{2}}, (\Lambda u)(\xi) = \frac{1}{\pi}\int\limits_{-1}^{1} \ln|\tau-\xi| \frac{u(\tau)d\tau}{\sqrt{1-\tau^2}}, \quad |\xi|<1; \tag{7}$$

$$K : L^2_{-\frac{1}{2}} \to L^2_{-\frac{1}{2}}, (Ku)(\xi) = \frac{1}{\pi}\int\limits_{-1}^{1} K(\xi,\tau) \frac{u(\tau)d\tau}{\sqrt{1-\tau^2}}, \quad |\xi|<1; \tag{8}$$

$$Q : L^2_{0,-\frac{1}{2}} \to L^2_{\frac{1}{2}}, (Qu)(\xi) = \frac{1}{\pi}\int\limits_{-1}^{1} Q(\xi,\tau) \frac{u(\tau)d\tau}{\sqrt{1-\tau^2}}, \quad |\xi|<1; \tag{9}$$

$$P : L^2_{-\frac{1}{2}} \to L^2_{-\frac{1}{2}}, (Pu)(\xi) = \sqrt{1-\xi^2}\cdot u(\xi), \quad |\xi|<1; \tag{10}$$

$$J : L^2_{0,-\frac{1}{2}} \to L^2_{\frac{1}{2}}, (Ju)(\xi) = \frac{1}{\pi}\int\limits_{-1}^{\xi} \frac{u(\tau)d\tau}{\sqrt{1-\tau^2}}, \quad |\xi|<1; \tag{11}$$

$$\Gamma : L^2_{0,-\frac{1}{2}} \to L^2_{\frac{1}{2}}, (\Gamma u)(\xi) = \frac{1}{\pi}\int\limits_{-1}^{1} \frac{1}{\tau-\xi} \frac{u(\tau)d\tau}{\sqrt{1-\tau^2}}, \quad |\xi|<1; \tag{12}$$

and the mappings

$$A : L^2_{0,-\frac{1}{2}} \to L^2_{\frac{1}{2}}; \quad A = \Gamma - c_1 J + Q; \tag{13}$$

$$B : L^2_{-\frac{1}{2}} \to L^2_{-\frac{1}{2}}, \quad B = I - c_2(P\Lambda) + K. \tag{14}$$

With the preceding notation (7)-(14), equation (1) with the additional condition (2) and equation (3) can be written as

$$A\vartheta = f; \tag{15}$$

$$B\nu = g. \tag{16}$$

**Proposition 1**

The operator $A : L^2_{0,-\frac{1}{2}} \to L^2_{\frac{1}{2}}$ is invertible and operator $A^{-1}$ is bounded.

**Proof.**

The operator A is the sum of invertible operator $\Gamma$ (see [6, 7] ) and a compact operator $-c_1 J + Q$. Hence, by virtue of Nikolsky criterion (see [8], p. 150), $Ind(A|_{L^2_{0,-\frac{1}{2}} \to L^2_{\frac{1}{2}}}) = 0$. From the uniqueness of the problem solution (1)-(2) follows, that $dim(ker(A|_{L^2_{0,-\frac{1}{2}} \to L^2_{\frac{1}{2}}})) = 0$ . Hence, $A(L^2_{0,-\frac{1}{2}}) = L^2_{\frac{1}{2}}$. Therefore, the operator A is bijective and bounded. So, by the corollary to the Banach Open Mapping Theorem operator $A$ has the bounded inverse.

**Proposition 2**

The operator $B : L^2_{-\frac{1}{2}} \to L^2_{-\frac{1}{2}}$ is invertible and operator $B^{-1}$ is bounded.

**Proof.**

The operator $P\Lambda$ is compact, as the composition of the bounded operator $P$ and the compact operator $\Lambda$. This follows from the compactness of the operator $-c_2 P\Lambda + K$ and the Fredholm theorem (see [8], p. 146) that $Ind(B|_{L^2_{-\frac{1}{2}} \to L^2_{-\frac{1}{2}}}) = 0$.

From the uniqueness of the problem solution (3) follows that

$$dim(ker(B|_{L^2_{-\frac{1}{2}} \to L^2_{-\frac{1}{2}}})) = 0.$$

Hence, $B(L^2_{-\frac{1}{2}}) = L^2_{-\frac{1}{2}}$. Therefore, the operator $B$ is bijective and bounded. So, by the corollary to the Banach Open Mapping Theorem the operator $B$ has the bounded inverse.

## 3. The statement of the problems for the approximate solution of the problems (1)-(2) and (3).

Let us consider sets of the points

$$t_k^n = \cos\left(\frac{2k-1}{2n}\pi\right), k = 1, \ldots, n; \quad t_{0,j}^n = \cos\left(\frac{j}{n}\pi\right), j = 1, \ldots, n-1. \tag{17}$$

Here the points $\{t_k^n\}_{k=1}^n$ are the zeros of Chebychev polynomials of the first kind $T_n(\tau)$ and the points $\left\{t_{0,j}^n\right\}_{j=1}^{n-1}$ are zeros of Chebychev polynomials of the second kind $U_{n-1}(\xi)$ .

We introduce the functions:

$$l_{1,n-1,k}(\xi) = \frac{1}{n}\left[1 + 2\sum_{p=1}^{n-1} T_p(\xi) \cdot T_p(\tau_k^n)\right], \quad k = 1, \ldots, n; \tag{18}$$

$$l_{2,n-2,j}(\xi) = \frac{U_{n-1}(\xi)}{U'_{n-1}(t^n_{0,j})(\xi - t^n_{0,j})} \quad j = 1, \ldots, n-1; \tag{19}$$

$$f_n(\xi) = \sum_{j=1}^{n-1} f(t^n_{0,j}) \cdot l_{2,n-2,j}(\xi); \tag{20}$$

$$g_n(\xi) = \sum_{k=1}^{n} g(t^n_k) \cdot l_{1,n-1,k}(\xi). \tag{21}$$

The functions $f_n(\xi)$ are Lagrange interpolation polynomials of the functions $f(\xi)$ with the interpolation points $\{t^n_{0,j}\}_{j=1}^{n-1}$ and the functions $g_n(\xi)$ are interpolating polynomials of the functions $g(\xi)$ with the interpolation points $\{t^n_k\}_{k=1}^n$. We introduce the operators:

$$Q_n : \Pi^0_{n-1,-\frac{1}{2}} \to \Pi_{n-2,\frac{1}{2}}, \quad (Q_n u)(\xi) = \frac{1}{\pi}\int_{-1}^{1} Q_n(\xi,\tau)\frac{u(\tau)d\tau}{\sqrt{1-\tau^2}}, \quad |\xi| < 1; \tag{22}$$

$$K_n : \Pi_{n-1,-\frac{1}{2}} \to \Pi_{n-1,-\frac{1}{2}}, \quad (K_n u)(\xi) = \frac{1}{\pi}\int_{-1}^{1} K_n(\xi,\tau)\frac{u(\tau)d\tau}{\sqrt{1-\tau^2}}, \quad |\xi| < 1; \tag{23}$$

$$P_n : \Pi_{n-1,-\frac{1}{2}} \to \Pi_{n-1,-\frac{1}{2}},$$

$$(P_n u)(\xi) = \sum_{k=1}^{n} \sqrt{1-(t^n_k)^2} \cdot u(t^n_k) \cdot l_{1,n-1,k}(\xi), \quad |\xi| < 1; \tag{24}$$

where

$$K_n(\xi,\tau) = \sum_{j=1}^{n}\sum_{k=1}^{n} K(t^n_j, t^n_k) \cdot l_{1,n-1,j}(\xi) \cdot l_{1,n-1,k}(\tau); \tag{25}$$

$$Q_n(\xi,\tau) = \sum_{j=1}^{n-1}\sum_{k=1}^{n} Q(t^n_{0,j}, t^n_k) \cdot l_{2,n-2,j}(\xi) \cdot l_{1,n-1,k}(\tau). \tag{26}$$

Note that the functions $(P_n u)(\xi)$ are Lagrange interpolation polynomials of the functions $(Pu)(\xi)$ with the interpolation points $\{t^n_{0,j}\}_{j=1}^{n-1}$.

Let us consider the integral operators:

$$A_n : \Pi^0_{n-1,-\frac{1}{2}} \to \Pi_{n-2,\frac{1}{2}}, \quad A_n = \Gamma - c_1 J + Q_n; \tag{27}$$

$$B_n : \Pi_{n-1,-\frac{1}{2}} \to \Pi_{n-1,-\frac{1}{2}}, \quad B_n = I - c_2(P_n\Lambda) + K_n \tag{28}$$

and equations

$$A_n \vartheta_n = f_n, \tag{29}$$

$$B_n \nu_n = g_n. \tag{30}$$

**Proposition 3**

For all natural numbers the following inequality holds:

$$\|A - A_n\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L^2_{\frac{1}{2}}} \leq \frac{M_1}{n^{\mu+\gamma}}. \tag{31}$$

Besides, $\|A - A_n\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L^2_{\frac{1}{2}}} \to 0$ as $n \to \infty$.

**Proof.**

The following estimate (see [6, 7]) holds:

$$\|Q - Q_n\|_{L^2_{-\frac{1}{2}} \to L^2_{\frac{1}{2}}} \leq \frac{M_2}{n^{\mu+\gamma}}. \tag{32}$$

This estimate is the consequence of Jackson's Theorems (see Corollary 1 of Th. 2 in [9], p.164 ). From (32) and equality

$$\|A - A_n\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L^2_{\frac{1}{2}}} = \|Q - Q_n\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L^2_{\frac{1}{2}}} \tag{33}$$

follow that the Proposition 3 is valid.

**Proposition 4**

The following inequality holds for all natural numbers:

$$\|B - B_n\|_{\Pi_{n-1,-\frac{1}{2}} \to L^2_{-\frac{1}{2}}} \leq \frac{M_3}{n^{\psi}}. \tag{34}$$

Hence, $\|B - B_n\|_{\Pi_{n-1,-\frac{1}{2}} \to L^2_{-\frac{1}{2}}} \to 0$ as $n \to \infty$.

**Proof.**

If the function $s(\xi)$ is the polynomial of degree $n$ then the the function $\sqrt{1-\xi^2} \cdot s(\xi)$ is the element of space $C^{0,\frac{1}{2}}[-1,1]$. From that fact and by corollary of Jackson's Theorems [9, p. 164] follows:

$$\|Pu - P_n u\|_{L^2_{-\frac{1}{2}}} \leq \frac{M_4}{\sqrt{n}}, \quad \forall u \in \Pi_{n,-\frac{1}{2}}. \tag{35}$$

Hence, the estimation (35) shows that the operators

$$W_n : \Pi_{n-1,-\frac{1}{2}} \to L^2_{-\frac{1}{2}}, \quad W_n = P - P_n$$

are bounded. Futhemore,

$$\|P - P_n\|_{\Pi_{n-1,-\frac{1}{2}} \to L^2_{-\frac{1}{2}}} \leq \frac{M_4}{\sqrt{n}}, \quad \forall n \in N. \tag{36}$$

The following inequality clearly holds

$$\|B-B_n\|_{\Pi_{n-1,-\frac{1}{2}}\to L^2_{-\frac{1}{2}}} \le |c_2|\cdot\|P-P_n\|_{\Pi_{n-1,-\frac{1}{2}}\to L^2_{-\frac{1}{2}}}\cdot\|\Lambda\|_{\Pi_{n-1,-\frac{1}{2}}\to L^2_{-\frac{1}{2}}}+$$

$$+\|K-K_n\|_{\Pi_{n-1,-\frac{1}{2}}\to L^2_{-\frac{1}{2}}}. \tag{37}$$

In [6, 7] the proof of inequalities

$$\|K-K_n\|_{L^2_{-\frac{1}{2}}\to L^2_{-\frac{1}{2}}} \le \frac{M_5}{n^\psi} \tag{38}$$

was given. From (36)-(38) follows the validity of proposition 4.

In the monograph [10, p.19] you can see the following theorem:

**Theorem 1**

Let $X$ and $Y$ be normed linear spaces and let $\widetilde{X}\subset X$ and $\widetilde{Y}\subset Y$ be finite-dimensional subspaces of the same dimension. We consider two equations.

The equation for exact solution of the problem

$$Au=f,\quad u\in X, f\in Y \tag{39}$$

and the equation for the approximate solution of the problem

$$A\widetilde{u}=f,\quad \widetilde{u}\in\widetilde{X},\widetilde{f}\in\widetilde{Y}. \tag{40}$$

Assume that:

1) the operator $A$ is invertible and the operator $A^{-1}$ is bounded,

2) the inequality holds

$$p=\|A^{-1}\|_{Y\to X}\cdot\|A-\widetilde{A}\|_{\widetilde{X}\to Y}<1. \tag{41}$$

Then

1) for any function $\widetilde{f}\in\widetilde{Y}$ the equation (40) has a unique solution $\widetilde{u}^*\in\widetilde{X}$ ;

2) let $u^*\in X$ be the solution of equation (39) and let $\delta=\|f-\widetilde{f}\|_Y$ then

$$\|u^*-\widetilde{u}^*\|_Y \le \|A^{-1}\|_{Y\to X}\cdot(1-p)^{-1}\cdot(\delta+p\|f\|_y). \tag{42}$$

Proposition 1-3 and Theorem 1 lead us to the following result.

**Theorem 2**

For all positive values exceeding the certain number $M$ the following statements hold true.

1) The problems (29) and (30) have a unique solution.

2) $\vartheta_n\in\Pi^0_{n-1,-\frac{1}{2}}$ and $\nu_n\in\Pi_{n-1,-\frac{1}{2}}$.

3) The sequence $\{\vartheta_n\}_{n=M}^{\infty}$ converges to the exact solution of the problem (15) in the norm of space $L^2_{\frac{1}{2}}$ . Moreover, $\|\vartheta-\vartheta_n\|_{L^2_{\frac{1}{2}}}\le\frac{M_6}{n^{\mu+\gamma}}$.

4) The sequence $\{\nu_n\}_{n=M}^{\infty}$ converges to the exact solution of the problem (16) in the norm of space $L^2_{-\frac{1}{2}}$ . Moreover, $\|\nu - \nu_n\|_{L^2_{\frac{1}{2}}} \leq \frac{M_7}{n^{\psi}}$.

**Proof.**

Let's define the numbers:

$$p_{1,n} = \|A^{-1}\|_{L^2_{\frac{1}{2}} \to L^2_{0,-\frac{1}{2}}} \cdot \|A - A_n\|_{\Pi^0_{n-1,-\frac{1}{2}} \to L^2_{\frac{1}{2}}}; \quad (43)$$

$$p_{2,n} = \|B^{-1}\|_{L^2_{-\frac{1}{2}} \to L^2_{-\frac{1}{2}}} \cdot \|B - B_n\|_{\Pi_{n-1,-\frac{1}{2}} \to L^2_{-\frac{1}{2}}}. \quad (44)$$

From the Propositions 3 and 4 and the existence of bounded operators $A^{-1}$ and $B^{-1}$ it follows the existence of number M that

$$(p_{1,n} < 1) \wedge (p_{2,n} < 1) \quad \forall n \in N, n > M. \quad (45)$$

Appealing to the Theorem 1 and (45) it concludes that

$$\|\vartheta_n - \vartheta\|_{L^2_{\frac{1}{2}}} \leq \|A^{-1}\|_{L^2_{\frac{1}{2}} \to L^2_{0,-\frac{1}{2}}} \cdot (1 - p_{1,n})^{-1} \cdot (\|f - f_n\|_{L^2_{\frac{1}{2}}} + p_{1,n}\|f\|_{L^2_{\frac{1}{2}}}), \quad (46)$$

$$\|\nu - \nu_n\|_{L^2_{-\frac{1}{2}}} \leq \|B^{-1}\|_{L^2_{-\frac{1}{2}} \to L^2_{-\frac{1}{2}}} \cdot (1 - p_{2,n})^{-1} \cdot (\|g - g_n\|_{L^2_{-\frac{1}{2}}} + p_{2,n}\|g\|_{L^2_{-\frac{1}{2}}}). \quad (47)$$

On using (46)-(47) and propositions 3-4 we complete the proof of Theorem 2.

## 3. Conclusions.

In the article [4], the discretization of equation (1) with the additional condition (2) and the equation (3) was performed by using the method of discrete singularities. As a result of discretization the systems of linear algebraic equations in the unknowns $\{\vartheta_n(t_k^n)\}_{k=1}^n$ and $\{\nu_n(t_k^n)\}_{k=1}^n$ had been obtained. These systems of linear algebraic equations are equivalent to the problems (29), (30). The proof of the equivalence follows from the one-to-one correspondence between the sets$\{\vartheta_n(t_k^n)\}_{k=1}^n$ , $\{\nu_n(t_k^n)\}_{k=1}^n$ and the polynomials $\vartheta_n(\tau)$, $\nu_n(\tau)$. Thus, finding the approximate numerical solution of the equation (1) with the additional condition (2) and the equation (3) is reduced to solving the system of linear algebraic equations. The convergence of the approximations to the exact solution is guaranteed by propositions proved in this article. Also, the rate of convergence of the approximate solutions to the exact solution had been found.

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